The Pythagoras number of function fields

Seminar on Arithmetic Geometry and Algebraic Groups

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24 April 2023

Sums of squares

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Every positive rational number is a sum of 4 squares of rational numbers.

We want to interpret these results as describing the value of an invariant associated to the rings $\mathbb{Z}$ and $\mathbb{Q}$.

Sums of squares

For a commutative ring K and $n \in \mathbb{N}$, define:

$$\Sigma_n K^2 = \{x_1^2 + \ldots + x_n^2 \mid x_1, \ldots, x_n \in K\}$$

$$\Sigma K^2 = \bigcup_{n\in\mathbb{N}} \Sigma_n K^2$$

$$p(K) = \inf\{n \in \mathbb{N} \mid \Sigma K^2 = \Sigma_n K^2\} \in \mathbb{N} \cup \{\infty\}.$$

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We call $p(K)$ the *Pythagoras number of* K. We will focus on the case where K is a field.

Examples and basic observations

Some first observations:

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- If char $K = 2$, then $x_1^2 + \ldots + x_n^2 = (x_1 + \ldots + x_n)^2$ for $x_1, \ldots, x_n \in K$. Hence $p(K) = 1$.

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- If K is a finite field with char $K \neq 2$, then $p(K) = 2$.

Examples and basic observations

- If $2 \in K^\times$ and $-1 \in \Sigma_n K^2$, then $K = \Sigma_{n+1} K^2$. In particular, $p(K) \leq n + 1$.

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If $-1 = x_1^2 + \ldots + x_n^2$ for some $x_1, \ldots, x_n \in K$, and if $a \in K$ is arbitrary, then

$$a = \left(\frac{a+1}{2}\right)^2 - \left(\frac{a-1}{2}\right)^2 = \left(\frac{a+1}{2}\right)^2 + \left(x_1 \frac{a-1}{2}\right)^2 + \ldots + \left(x_n \frac{a-1}{2}\right)^2.$$

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We call a field K *real* if $-1 \notin \Sigma K^2$ and *nonreal* if $-1 \in \Sigma K^2$.

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  In general, for $n \geq 2$, $n + 2 \leq p(\mathbb{R}(X_1, \ldots, X_n)) \leq 2^n$,

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- For $n \geq 2$, $n + 4 \leq p(\mathbb{Q}(X_1, \ldots, X_n)) \leq 2^{n+1}$ (Coilliot-Thélène; Jannsen, 2016)

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We also have the following:

Theorem (Hoffmann, 1999)

For every $n \in \mathbb{N}^+ \cup \{\infty\}$, there exists a real field K with $p(K) = n$.

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Does there exist a finite field extension L/K with $p(L) > p(K) + 2$?

In fact, we do not know of any example of a field K where $p(K(X)) > p(K) + 2$, but we nevertheless cannot answer the following:

Question

If K is a field with $p(K) < \infty$, do we have $p(K(X)) < \infty$?

Growth of the Pythagoras number

Pythagoras numbers of finite field extensions and of function fields are related:

Theorem (Pfister; Becher, Van Geel, 2009)

Let K be a real field, $n \in \mathbb{N}$. The following are equivalent:

1. $p(K(X)) \leq 2^{n+1}$,
2. $p(L) < 2^{n+1}$ *for every real finite field extension* L/K,
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Theorem (Becher, D., Grimm, Manzano-Flores, Zaninelli, 2023)

Let K be a field with $p(K(X)) = 2$. Then $p(L) \leq 5$ for all finite field extensions $L/K(X)$. In particular, $p(K(X,Y)) \leq 8$.

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3. **Hereditarily Pythagorean fields:**
   - How to characterise for a real field K that $p(K(X)) = 2$?
   - Examples and constructions of such fields.

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   - Examples and constructions of such fields.
4. **Valuations and local-global principles:**
   - Characterising her. pyth. fields with valuations.
   - Bounds on $p(L)$ for $L/K(X)$ finite based on a local-global principle for quadratic forms.

Pfister forms

Let always K be a field, char $K \neq 2$.
For $n \in \mathbb{N}$ and $a_1, \ldots, a_n \in K^\times$, we define inductively the *n-fold Pfister form* $\langle\langle a_1, \ldots, a_n \rangle\rangle_K$ as a quadratic form in 2^n variables:

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- $n > 0$: $\langle\langle a_1, \ldots, a_n \rangle\rangle_K(X_1, \ldots, X_{2^n}) =$
$\langle\langle a_1, \ldots, a_{n-1} \rangle\rangle_K(X_1, \ldots, X_{2^{n-1}}) - a_n \langle\langle a_1, \ldots, a_{n-1} \rangle\rangle_K(X_{2^{n-1}+1}, \ldots X_{2^n})$.

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We will say that K *satisfies* I_n if every $(n+1)$-fold Pfister form over K has a non-trivial zero.
We say that K *satisfies* I_n^* if every finite field extension L/K satisfies I_n.

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Examples (always characteristic not 2):

- algebraically closed fields satify I_0^*,
- finite fields satisfy I_1^*,
- non-real global fields satisfy I_2^* (e.g. by Hasse-Minkowski).

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Proof idea.

- As a consequence to Voevodsky's solution to the Milnor Conjecture (2003), we have that

$$K \text{ satisfies } I_n^* \quad \Leftrightarrow \quad \mathrm{cd}_2(K) \leq n$$

where $\mathrm{cd}_2(K)$ denotes the 2-cohomological dimension of K.

- It is a standard result from Galois cohomology that $\mathrm{cd}_2(K(X)) \leq \mathrm{cd}_2(K) + 1$.

□

The property I_n and Pythagoras numbers

Theorem (Pfister, Elman-Lam, '90ies)

If $K(\sqrt{-1})$ satisfies I_n, then $p(K) \leq 2^n$.

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Applications:

- (Pfister, 1970) $p(\mathbb{R}(X_1, \ldots, X_n)) \leq 2^n$,
- $p(\mathbb{Q}(X_1, \ldots, X_n)) \leq 2^{n+2}$.

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Conclusion: the well-behavedness of $p(K(X_1, \ldots, X_n))$ for $K = \mathbb{R}$ or $K = \mathbb{Q}$ comes from the well-behavedness of the property I_n^*.
What if K does not satisfy I_n^* for any n?

Pythagorean and euclidean fields

We call a real field K *pythagorean* if $p(K) = 1$. We call a real field K *hereditarily pythagorean* if $p(L) = 1$ for every finite real extension L/K.

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Equivalently (see Theorem earlier on), a real field K is hereditarily pythagorean if $p(K(X)) = 2$. We want to prove:

Theorem (Becher, D., Grimm, Manzano-Flores, Zaninelli, 2023)

Let K be a hereditarily pythagorean field. Then $p(L) \leq 5$ for all finite field extensions $L/K(X)$. In particular, $p(K(X, Y)) \leq 8$.

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We will first look at some special cases.

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If K is hereditarily euclidean, then K is hereditarily pythagorean. Furthermore, one has that $K(\sqrt{-1})(X, Y)$ is I_2^*, hence $p(L) \leq 3$ for any finite field extension $L/K(X)$.

Formal Laurent series

For a field K, we may consider the field of formal Laurent series

$$K((T)) = \left\{ \sum_{j=N}^{+\infty} a_i T^i \;\middle|\; N \in \mathbb{Z}, a_i \in K \right\}.$$

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Proposition

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Proof idea.

Let $L/K((T))$ be a real finite field extension.

- L carries a complete discrete valuation v whose residue field Lv is a finite extension of K, hence pythagorean.
- By Hensel's Lemma, also L is pythagorean.

□

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- $K_n = \mathbb{R}((T_1))\dots((T_n))$ is hereditarily pythagorean, but $K_n(\sqrt{-1}) = \mathbb{C}((T_1))\dots((T_n))$ is not I_m for $m < n$.

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- Consequently, $K = \bigcup_{n\in\mathbb{N}} K_n$ is hereditarily pythagorean, but not I_m for any $m \in \mathbb{N}$.
- Nevertheless, in the above setup, (Becher, Grimm, Van Geel, 2012) $p(K(X,Y)) \leq 4$.

Hereditarily half-euclidean fields

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We call a field K *half-euclidean* if it is pythagorean and has exactly two orderings. We call a real field K *hereditarily half-euclidean* if every real finite field extension is half-euclidean.

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Example: Let $\leq_1$ and $\leq_2$ be the two orderings on $\mathbb{Q}(\sqrt{2})$. By Zorn's Lemma, there exists a maximal algebraic extension K of $\mathbb{Q}(\sqrt{2})$ subject to the condition that $\leq_1$ and $\leq_2$ extend to K. Then K is hereditarily half-euclidean.

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If K is hereditarily half-euclidean, then $K(\sqrt{-1})$ is I_1^*, hence $p(L) \leq 4$ for all finite field extensions $L/K(X)$.

Hereditarily half-euclidean fields

In summary, we have the following examples/constructions of hereditarily pythagorean fields:

- hereditarily euclidean fields,
- hereditarily half-euclidean fields,
- if K is hereditarily pythagorean, then so is $K((T))$.

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- hereditarily half-euclidean fields,
- if K is hereditarily pythagorean, then so is $K((T))$.

In fact, we will see now that this *almost* covers all examples ...

Henselian valuations

For a field K:

- A *valuation on* K is a group epimorphism

$$v : K^\times \twoheadrightarrow vK$$

where vK is an ordered abelian group (called the *value group of* v), such that $v(x+y) \geq \max\{v(x), v(y)\}$ for $x, y \in K$ with $x + y \neq 0$.

Henselian valuations

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- A *valuation on* K is a group epimorphism

$$v : K^\times \twoheadrightarrow vK$$

  where vK is an ordered abelian group (called the *value group of* v), such that $v(x+y) \geq \max\{v(x), v(y)\}$ for $x, y \in K$ with $x + y \neq 0$.
- Given a valuation v on K, we call:
  - $\mathcal{O}_v = \{x \in K^\times \mid v(x) \geq 0\} \cup \{0\}$ the *valuation ring of* v,
  - $\mathfrak{m}_v = \{x \in K^\times \mid v(x) > 0\} \cup \{0\}$ the *valuation ideal of* v,
  - $Kv = \mathcal{O}_v / \mathfrak{m}_v$ the *residue field of* v,
  - the number $\mathrm{rk}(v)$ of proper convex subgroups of vK the *rank of* v.

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  - $Kv = \mathcal{O}_v / \mathfrak{m}_v$ the *residue field of* v,
  - the number $\text{rk}(v)$ of proper convex subgroups of vK the *rank of* v.
- We call a valuation v on K
  - *complete* if the induced field topology is complete,
  - *henselian* if v extends uniquely to every finite field extension of K.

Henselian valuations

Example: let K_0 be any field, consider the field $K = K_0((T))$ and define the map

$$v_T : K^\times \to \mathbb{Z} : \sum_{j=N}^{+\infty} a_j T^j \mapsto \min\{j \in \mathbb{Z} \mid a_j \neq 0\}.$$

This is a complete, henselian valuation of rank 1 with residue field K_0.

Henselian valuations

Proposition

Let K be a field, v a henselian valuation on K with char $Kv \neq 2$. *Then one of the following occurs:*

- $p(K) = p(Kv)$,
- $p(K) = p(Kv) + 1$, $-1 \in \Sigma_{p(K)} K^2$ *and* $-1 \in \Sigma_{p(K)} (Kv)^2$.

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In particular, we obtain a generalisation of the observation that if K is hereditarily pythagorean, then so is $K((T))$:

Corollary

Let K be a field, v a henselian valuation on K. If Kv is hereditarily pythagorean, then so is K.

Henselian valuations

The promised classification of hereditarily pythagorean fields:

Theorem (Bröcker, 1976)

A field K is hereditarily pythagorean if and only if there exists a henselian valuation v on K whose residue field is hereditarily euclidean or hereditarily half-euclidean.

Mehmeti's local-global principle

Theorem (Mehmeti, 2019)

Let K be a field with $\operatorname{char} K \neq 2$ *and with a complete rank 1 valuation v. Let $L/K(X)$ be a finite field extension. Let $q(X_1, \ldots, X_n)$ be a quadratic form over L for some $n \geq 3$.*
If q has a non-trivial zero over every completion L^w where w is a rank 1 valuation on L with either $w|_K = 0$ or $w|_K = v$, then q has a non-trivial zero over L.

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In particular:

Corollary

Let K be a field with a complete rank 1 valuation v. Let $L/K(X)$ be a finite field extension. We have

$$p(L) = \sup\{p(L^w) \mid w \text{ rank 1 valuation on } F \text{ with } w|_K = 0 \text{ or } w|_K = v\}.$$

Proof of main theorem

To prove

Theorem (Becher, D., Grimm, Manzano-Flores, Zaninelli, 2023)

Let K be a hereditarily pythagorean field. Then $p(L) \leq 5$ for all finite field extensions $L/K(X)$. In particular, $p(K(X,Y)) \leq 8$.

note that we have the following:

- The theorem holds when K is hereditarily euclidean.
- The theorem holds when K is hereditarily half-euclidean.
- (Bröcker) In general, K carries a henselian valuation whose residue field is hereditarily euclidean of hereditarily half-euclidean.

Proof of main theorem

We are thus done if we prove the following:

Theorem

Let $n \in \mathbb{N}$ not a power of 2 and consider the following property for a field K:

$$\text{For every field field extension } L/K(X) \text{ we have } p(L) \leq n. \tag{1}$$

If a field K carries a henselian valuation v whose residue field Kv satisfies (1) and char $Kv \neq 2$*, then K satisfies (1).*

Proof of main theorem

Proof idea.

We prove this in multiple steps:

1. *v is complete of rank one*: This case follows from Mehmeti's local-global principle.

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3. *v is henselian of finite rank*: Apply induction on the rank.

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Proof of main theorem

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1. *v is complete of rank one*: This case follows from Mehmeti's local-global principle.
2. *v is henselian of rank one*: Mehmeti's local-global principle can be extended to this situation by standard arguments.
3. *v is henselian of finite rank*: Apply induction on the rank.
4. *v is henselian of arbitrary rank*: By a limit argument, reduce to considering everything over some subfield of K of finite transcendence degree and hence finite rank.

□

Outlook

What's next?

Conjecture

If K is a field and $p(K(X)) < \infty$, then $p(K(X,Y)) < \infty$.

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If K is a field and $p(K(X)) < \infty$, then $p(K(X,Y)) < \infty$.

- Can we show $p(K(X)) \leq 4 \Rightarrow p(K(X,Y)) < \infty$?
- Can we show $p(K(X)) = 2 \Rightarrow p(K(X,Y,Z)) < \infty$?
- Can we find a more elementary proof that $p(K(X)) = 2 \Rightarrow p(K(X,Y)) \leq 8$?

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Conjecture

If $p(K(X)) = 2$, then $p(L) \leq 3$ for every finite field extension $L/K(X)$. In particular, $p(K(X,Y)) \leq 4$.

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Conjecture

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Conjecture

If $p(K(X)) = 2$, then $p(L) \leq 3$ for every finite field extension $L/K(X)$. In particular, $p(K(X,Y)) \leq 4$.

In view of the previous theorem, in order to prove this conjecture, it suffices to prove it when K is hereditarily half-euclidean.

[1] K.J. Becher, N. Daans, D. Grimm, G. Manzano-Flores, M. Zaninelli.
The Pythagoras number of a Rational Function Field in two variables.
preprint available as `arXiv:2302.11423` (2023).

[2] K.J. Becher, J. Van Geel.
Sums of squares in function fields of hyperelliptic curves
Math. Z. 261 (2009), 829-844.

[3] L. Bröcker.
Characterization of Fans and Hereditarily Pythagorean Fields.
Math. Z. 151 (1976): 149-164.

[4] D.W. Hoffmann.
Pythagoras number of fields
J. Am. Math. Soc. 12 (1999): 839-848.

[5] U. Jannsen.
Hasse principles for higher-dimensional fields.
Ann. of Math. 183 (2016), 1-71.

[6] V. Mehmeti.
Patching over Berkovich curves and quadratic forms.
Comp. Math. 155 (2019), 2399-2438.

[7] A. Pfister.
Quadratic forms with applications to algebraic geometry and topology.
London Mathematical Society Lecture Note Series 217, Cambridge University Press, 1995.

[8] V. Voevodsky.
Motivic cohomology with $\mathbb{Z}/2$-coefficients.
Publ. Math. Inst. Hautes Études Sci. 98 (2003), 59-104.

Thanks for your attention!

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