Existential definitions: from local to global fields

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Places on global fields

A *global field* is a field K which is either

- a finite extension of $\mathbb{Q}$, or
- a finite extension of $\mathbb{F}_p(T)$ for some prime number p.

In the first case we call K a *number field*, in the second case a *global function field*.

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Given a field K, we call the embeddings into $\mathbb{R}$ and completions *places* of K, indexed by a set $\mathbb{P}'(K)$ (or simply $\mathbb{P}'$ if the field is clear from context). Embeddings into $\mathbb{R}$ are also called *archimedean*, *infinite* or *real* places, whereas completions are also called *non-archimedean* or *finite* places. Let us index the set of finite places by $\mathbb{P}(K)$ (or $\mathbb{P}$ if the field is clear from context). For $\mathfrak{p} \in \mathbb{P}'(K)$ we denote by $K_{\mathfrak{p}}$ the local field (or $\mathbb{R}$) into which $\mathfrak{p}$ is an embedding; by abuse of notation we consider K to be a subfield of $K_{\mathfrak{p}}$.

Places on global fields

Theorem 1.1 (Hasse-Minkowski)

Let K be a global field, Q a quaternion algebra defined over K. Then Q is split if and only if $Q_{K_\mathfrak{p}}$ is split for every $\mathfrak{p} \in \mathbb{P}'(K)$.

E.g. over $\mathbb{Q}$, a quaternion algebra Q is split if and only if it is split over $\mathbb{R}$ and $\mathbb{Q}_p$ for every prime number p.

We write $Q_\mathfrak{p}$ as shorthand for $Q_{K_\mathfrak{p}}$.

Recap: S-sets

Recall: for a quaternion algebra Q over a field K, define

$$S(Q) = \{\mathrm{Trd}(x) \mid x \in Q \setminus K, \mathrm{Nrd}(x) = 1\} \subseteq K.$$

This set is existentially definable in the language of rings $\mathcal{L}$, uniformly in the parameters defining Q. More precisely:

Proposition 1.2

Let $\varphi(x, a, b)$ be $\mathcal{L}$-formula

$$\exists z_1, z_2, z_3 : 1 \doteq az_1^2 + bz_2^2 + (4 - x^2)abz_3^2.$$

Then for every field K with $\mathrm{char}(K) \neq 2$ and $a, b \in K^\times$ we have

$$S((a, b)_K) = \{x \in K \mid K \models \varphi(x, a, b)\}.$$

Note: we might write $S(a, b)$ instead of $S((a, b)_K)$.

Recap: S-sets

Some previously discussed properties:

Proposition 1.3

1. *$S(Q) = K \Leftrightarrow Q$ is split.*
2. *If K is a local field with valuation ring $\mathcal{O}$ and Q is the unique non-split quaternion algebra over K, then*
   1. $S(Q) + S(Q) = \mathcal{O}$,
   2. *$S(Q)$ is open with respect to the v-adic topology.*
3. *If $K = \mathbb{R}$ and Q is non-split, then $S(Q) =]-2, 2[$.*
4. *If K is a global field, then*

$$S(Q) = \bigcap_{\mathfrak{p} \in \mathbb{P}'(K)} S(Q_{\mathfrak{p}}) \cap K.$$

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$$S(Q) = \bigcap_{\mathfrak{p} \in \mathbb{P}'(K)} S(Q_{\mathfrak{p}}) \cap K.$$

Suppose now that K is a global field and Q a quaternion algebra defined over K. What is $S(Q) + S(Q)$?

Ramification sets

For a quaternion algebra Q over a global field K, define its *ramification set* as follows:

$$\Delta Q = \{\mathfrak{p} \in \mathbb{P}'(K) \mid Q_{K_\mathfrak{p}} \text{ is non-split}\}$$

Note: the Hasse-Minkowski theorem says precisely that Q is split if and only if $\Delta Q = \emptyset$.

$S(Q) + S(Q)$

For a global field K and a quaternion algebra Q defined over K, define

$$T(Q) = \bigcap_{\mathfrak{p} \in \Delta Q} \mathcal{O}_{\mathfrak{p}}$$

where $\mathcal{O}_{\mathfrak{p}}$ is the valuation ring corresponding to $\mathfrak{p}$ if it is a finite place, or the preimage of the interval $]-4, 4[$ if it is an infinite place. As before, for $a, b \in K^{\times}$, we might write $T(a, b)$ instead of $T((a, b)_K)$.

Theorem 1.4 (Poonen, Koenigsmann, 2010)

Let K be a global field, Q a quaternion algebra defined over K. One has

$$S(Q) + S(Q) = T(Q).$$

Outline

Plan for today:

- Give a proof of the Poonen-Koenigsmann Theorem, derive existential predicate, finish the proof of universal definability of $\mathbb{Z}$ in $\mathbb{Q}$.

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- Give a proof of the Poonen-Koenigsmann Theorem, derive existential predicate, finish the proof of universal definability of $\mathbb{Z}$ in $\mathbb{Q}$.
- Discuss generalisations of universal definability of $\mathbb{Z}$ in $\mathbb{Q}$ to other (global and non-global) fields.

Topologies induced by places

Let K be a field. For each place $\mathfrak{p}$, there is a natural topology on $K_\mathfrak{p}$ (called the $\mathfrak{p}$-topology) making $K_\mathfrak{p}$ into a topological field:

- If $\mathfrak{p}$ is an embedding $K \to \mathbb{R}$, then the $\mathfrak{p}$-topology is the Euclidean topology on $\mathbb{R}$.

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- If $\mathfrak{p}$ is an embedding $K \to \mathbb{R}$, then the $\mathfrak{p}$-topology is the Euclidean topology on $\mathbb{R}$.
- For every $\mathbb{Z}$-valuation v on K, the topology on the completion K_v has basic open sets

$$\{x \in K_v \mid v(x - x_0) > n\}$$

for $x_0 \in K_v$, $n \in \mathbb{N}$.

Artin-Whaples

Theorem 2.1 (Artin-Whaples)

Let K be a field. Endowing $\prod_{\mathfrak{p}\in\mathbb{P}'(K)} K_{\mathfrak{p}}$ with the product topology, K is a dense subspace.

When $\mathbb{P}'(K)$ consists only of finite places, this theorem follows from the so-called *Weak Approximation Theorem* from valuation theory.

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Lemma 2.2

Let K be a field, $S \subseteq \mathbb{P}'(K)$ a finite set. For each $\mathfrak{p} \in S$, let $A_\mathfrak{p}$ and $B_\mathfrak{p}$ be subsets of $K_\mathfrak{p}$ which are open w.r.t. the $\mathfrak{p}$-topology on $K_\mathfrak{p}$. Then

$$\bigcap_{\mathfrak{p}\in S} A_\mathfrak{p} \cap K + \bigcap_{\mathfrak{p}\in S} B_\mathfrak{p} \cap K = \bigcap_{\mathfrak{p}\in S}(A_\mathfrak{p} + B_\mathfrak{p}) \cap K.$$

Proof:

$S(Q) + S(Q)$

Proof of Theorem 1.4:

Existential predicate

The theorem tells us that for all $a, b \in K^\times$ we have

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Corollary 2.3

There exists an existential $\mathcal{L}$-formula ψ in 3 free variables such that for all $a, b \in K^\times$ we have

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This was the missing piece in the proof of the universal definability of $\mathbb{Z}$ in $\mathbb{Q}$ in last talk.

Universal definability in global fields

Theorem 3.1 (Koenigsmann, 2010)

$\mathbb{Z}$ *has a universal first-order definition in* $\mathbb{Q}$.

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Theorem 3.2 (Park, 2012)

Let K be a number field. The ring of integers $\mathcal{O}_K$ has a universal $\mathcal{L}_K$-definition in K.

Rings of S-integers

K a global field, $S \subseteq \mathbb{P}(K)$ finite. The *ring of S-integers* is defined to be

$$\mathcal{O}_S = \bigcap_{\mathfrak{p} \in \mathbb{P}(K) \setminus S} \mathcal{O}_{\mathfrak{p}}$$

where $\mathcal{O}_{\mathfrak{p}}$ is the valuation ring corresponding to the finite place $\mathfrak{p}$.

Examples:

- K a number field. Then $\mathcal{O}_{\emptyset} = \mathcal{O}_K$. In particular, if $K = \mathbb{Q}$, $\mathcal{O}_{\emptyset} = \mathbb{Z}$.

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- K a global function field with field of constants $\mathbb{F}_q$. $\mathcal{O}_\emptyset = \mathbb{F}_q$.
- $K = \mathbb{F}_q(T)$ with $\mathbb{F}_q$ a finite field, $S = \{v_\infty\}$. Then $\mathcal{O}_S = \mathbb{F}_q[T]$.

Rings of S-integers

Theorem 3.3 (Eisenträger and Morrison, 2017)

Let K be a global function field of odd characteristic, S a finite set of $\mathbb{Z}$-valuations on K. Then $\mathcal{O}_S$ has a universal $\mathcal{L}_K$-definition in K.

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The original proofs of Koenigsmann, Park, Eisenträger and Morrison rely on dividing the primes up in several ray classes and using fancier results from class field theory. By relying more on the use of Hilbert Reciprocity, one can combine and extend the proofs.

Theorem 3.4 (Daans, 2018)

Let K be a global field, S a finite set of $\mathbb{Z}$-valuations on K. Then $\mathcal{O}_S$ has a universal $\mathcal{L}_K$-definition in K.

Finitely generated subrings

Rings of S-integers of a global field K (for some finite set $S \subseteq \mathbb{P}(K)$) are precisely the finitely generated, integrally closed subrings of K with K as their fraction field. We can extend the result to non-integrally closed subrings:

Corollary 3.5

Let K be a global field, $R \subseteq K$ a finitely generated subring with $K = \mathrm{Frac}(R)$. Then R has a universal $\mathcal{L}_K$-definition in K.

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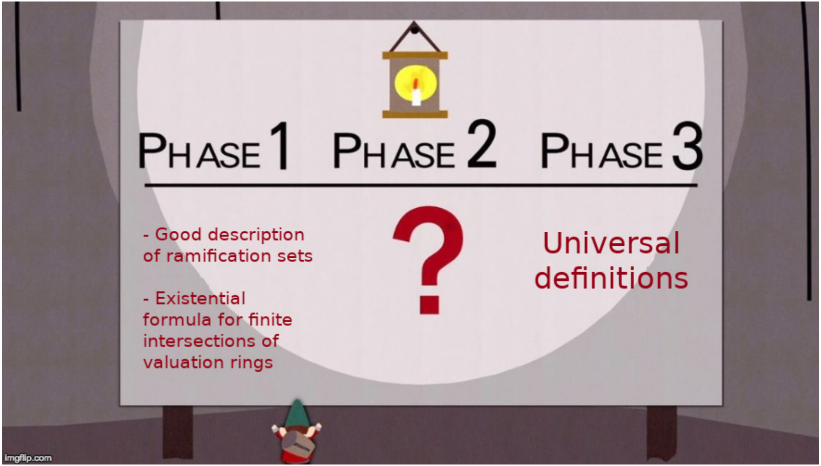
- that we have an existential $\mathcal{L}$-formula associating to (the parameters defining) a quaternion algebra a finite intersection of valuation rings, indexed by the ramification set of the quaternion algebra (Corollary 2.3).
- a good description of the ramification set, in particular Hilbert's Reciprocity law (that $\Delta(a, b)$ always contains an even number of elements)

Proof dissection



Outlook: universal definability in function fields over $\mathbb{Q}_p$

Consider the case where F is an algebraic function field over a local field K. Let $\mathcal{V}$ be the set of $\mathbb{Z}$-valuations which are trivial on $K^\times$. We can do something similar to obtain universal definability of rings of S-integers for some finite subset $S \subseteq \mathcal{V}$.

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- 3-fold Pfister forms (octonion algebras) replace quaternions
- The ramification set

  $$\Delta(a, b, c) = \{v \in \mathcal{V} \mid \langle\langle a, b, c\rangle\rangle_{F_v} \text{ is anisotropic}\}$$

  satisfies a Reciprocity Law, i.e. its image consists precisely of the finite subsets of $\mathcal{V}$ of even cardinality.

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  satisfies a Reciprocity Law, i.e. its image consists precisely of the finite subsets of $\mathcal{V}$ of even cardinality.
- There is an existential formula associating to $(a,b,c) \in (F^\times)^3$ the subring $\bigcap_{v \in \Delta(a,b,c)} \mathcal{O}_v$. This is the topic of my next two talks.

References

[Daa20] Nicolas Daans. "Universally defining finitely generated subrings of global fields". https://arxiv.org/abs/1812.04372. Mar. 2020.

[Dit18] Philip Dittmann. "Irreducibility of polynomials over number fields is diophantine". In: *Composito Mathematica.* 154 (2018), pp. 761–772. DOI: 10.1112/S0010437X17007977.

[EM18] Kirsten Eisenträger and Travis Morrison. "Universally and existentially definable subsets of global fields". In: *Math. Res. Lett.* 25.4 (2018), pp. 1173–1204.

[Koe16] Jochen Koenigsmann. "Defining $\mathbb{Z}$ in $\mathbb{Q}$". In: *Annals of Mathematics.* 183 (2016), pp. 73–93.

[Poo09] Bjorn Poonen. "Characterizing integers among rational numbers with a universal-existential formula". In: *Amer. J. Math.* 131 (2009), pp. 675–682.

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