Valuations on function fields

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Existential definitions and valuations

- Recall: to find an existential definition of valuation rings of $\mathbb{F}_p(T)$ (and later a universal definition of $\mathbb{F}_p[T]$) we used the existence of a local-global principle with respect to the discrete valuations on $\mathbb{F}_p(T)$.

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Sometimes, even when we want to talk only about a restricted set of valuations on a field, it is necessary to consider a larger class of valuations.

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- discuss existential definability of the field of constants in function fields.

Condition for isotropy over completion

Proposition 2.1

Let K be a field with $\mathrm{char}(K) \neq 2$, v a $\mathbb{Z}$-valuation on K. Let $q(X_1, \ldots, X_k) = \sum_{i=1}^{k} b_i X_i^2$ be a quadratic form defined over K for some $b_1, \ldots, b_k \in K$. The following are equivalent:

1. *q is isotropic over the completion K_v.*
2. *There exist $x_1, \ldots, x_k \in K$ such that*

$$v(q(x_1, \ldots, x_k)) > \min_{i=1}^{k} v(b_i x_i^2) + v(4).$$

Proof:

Coarsening

Let F be a field, K a subfield, w a valuation on F such that $wK^\times \neq 0$. Consider the following subset of F:

$$\mathcal{O}_w^{(K)} = \{ax \mid a \in K, x \in \mathfrak{m}_w\}.$$

Proposition 2.2

$\mathcal{O}_w^{(K)}$ is the smallest valuation ring on F containing $\mathcal{O}_w$ and K.

An valuation w' which has $\mathcal{O}_w^{(K)}$ as its valuation ring is called a *K-coarsening* of w. It is determined uniquely up to equivalence.

Coarsening

Proof:

The space of valuations

Let K be a field. Denote by $\mathcal{V}$ (Sylvy's $S_{\mathrm{val}}(K)$) the set of all valuations on K *up to equivalence* (or equivalently, the set of all valuation rings on K). We endow $\mathcal{V}$ with a topology called the *constructible topology* with basic clopen sets

$$\{v \in \mathcal{V} \mid v(x) \geq 0\}$$

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Proposition 2.3

$\mathcal{V}$ is compact with respect to the constructible topology.

Proof idea: Endow $\mathcal{P}(K)$ with the product topology. By Tychonoff's theorem (Axiom of Choice!) this space is compact. Show that the natural injection

$$\mathcal{V} \to \mathcal{P}(K) : v \mapsto \mathcal{O}_v$$

is a closed immersion.

Valuations on function fields

Let for the rest of the talk F be an algebraic function field over a local field (K, v) of characteristic different from 2. Let $\mathcal{V}$ be the space of valuations of F, $\mathcal{V}_0$ the subspace of valuations trivial on K. Fix a uniformiser π of v and let $\mathcal{V}_\pi$ be the subspace of $\mathcal{V}$ of *discrete* valuations w for which $w(\pi) > 0$.
For a quadratic form q defined over F, define the following sets:

$$\Delta_0 q = \{w \in \mathcal{V}_0 \mid q \text{ anisotropic over } F_w\}$$
$$\Delta_\pi q = \{w \in \mathcal{V}_\pi \mid q \text{ anisotropic over } F_w\}$$

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Proposition 3.1 (Kato, 1986)

Let q be a three-fold Pfister form defined over F. If $\Delta_\pi q = \emptyset$, then q is isotropic over F.

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Remark: if $w \in \mathcal{V}_\pi$, then $\mathcal{O}_w \cap K = \mathcal{O}_v$.

Passing between $\Delta_0 q$ and $\Delta_\pi q$

Proposition 3.2

Let $x \in \bigcap_{w \in \Delta_0 q} \mathcal{O}_w$, where q is a quadratic form defined over F. Then there exists an $a \in K^\times$ such that $ax \in \bigcap_{w \in \Delta_\pi q} \mathfrak{m}_w$.

Proof:

Defining fields of constants

Assume now additionally that F/K is a regular extension (without loss of generality if $\mathrm{char}(K) = 0$). Let $\mathcal{L}$ be the language of rings.

Proposition 3.3

K has an existential $\mathcal{L}$-definition in F. More specifically, for $d \in \mathbb{N}$ sufficiently large, we have for all $t \in F$ that

$$t \in K \quad \Leftrightarrow \quad F \models \exists y, x_1, x_2 (1 \doteq y^2(x_1^d - x_1) \wedge t^2 \doteq y^2(x_2^d - x_2)).$$

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Consider the following (hyper)elliptic curve, for $d \in \mathbb{N}$, $d \geq 3$:

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Lemma 3.4

Denote by $\mathfrak{m}_v$ the maximal ideal of the complete valuation ring on K. We have for all $d \geq 3$

$$\mathfrak{m}_v \subseteq \{y \in K \mid \exists x \in K : y^2 = x^d - x\}.$$

Proof:

Defining fields of constants

If $g \in \mathbb{N}$ and $d = 2g + 3$, the irreducible plane curve E_d has genus $g + 1$.

Proposition 3.5

Let C be an irreducible plane curve defined over K. If the genus of C is greater than the genus of F, then all F-rational points of C are K-rational.

Proof:

Defining fields of constants

Proof of Proposition 3.3:

Defining fields of constants

Koenigsmann showed that, for a function field in one variable F over a field K, it suffices to assume that K is *large* (in the sense of F. Pop) to obtain existential $\mathcal{L}$-definability of the field of constants. Examples of large fields:

- henselian valued fields,
- algebraically closed, real closed, or separably closed fields.

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