Defining subrings in p-adic function fields

Nicolas Daans

University of Antwerp

July 10, 2020

Recap

Let $\mathcal{L}$ be the language of rings. Let (K, v) be a local field with $\mathrm{char}(K) \neq 2$, F an algebraic function field over K. Let $\mathcal{V}$ be the space of valuations of F, $\mathcal{V}_0$ the subspace of valuations trivial on K. Fix a uniformiser π of v and let $\mathcal{V}_\pi$ be the subspace of $\mathcal{V}$ of *discrete* valuations w for which $w(\pi) > 0$.

For a quadratic form q defined over F, define the following sets:

$$\Delta_0 q = \{w \in \mathcal{V}_0 \mid q \text{ anisotropic over } F_w\}$$
$$\Delta_\pi q = \{w \in \mathcal{V}_\pi \mid q \text{ anisotropic over } F_w\}$$

Recap

Recall the following from previous talk:

Proposition 1.1 (Kato, 1986)

Let q be a three-fold Pfister form defined over F. If $\Delta_\pi q = \emptyset$, then q is isotropic over F.

Recap

Recall the following from previous talk:

Proposition 1.1 (Kato, 1986)

Let q be a three-fold Pfister form defined over F. If $\Delta_\pi q = \emptyset$, then q is isotropic over F.

Proposition 1.2

Let $x \in \bigcap_{w \in \Delta_0 q} \mathcal{O}_w$, where q is a quadratic form defined over F. Then there exists an $a \in K^\times$ such that $ax \in \bigcap_{w \in \Delta_\pi q} \mathfrak{m}_w$.

Recap

Recall the following from previous talk:

Proposition 1.1 (Kato, 1986)

Let q be a three-fold Pfister form defined over F. If $\Delta_\pi q = \emptyset$, then q is isotropic over F.

Proposition 1.2

Let $x \in \bigcap_{w \in \Delta_0 q} \mathcal{O}_w$, where q is a quadratic form defined over F. Then there exists an $a \in K^\times$ such that $ax \in \bigcap_{w \in \Delta_\pi q} \mathfrak{m}_w$.

Proposition 1.3

K has an existential $\mathcal{L}$-definition in F.

Statement & outline

Today, we give a proof of the following:

Theorem 1.4

There exists an existential $\mathcal{L}$-formula φ in 4 free variables such that for all $a, b, c \in F^\times$ one has

$$\bigcap_{v \in \Delta_0\langle\langle a,b,c\rangle\rangle} \mathcal{O}_v = \{x \in F^\times \mid F \models \varphi(x, a, b, c)\}.$$

Statement & outline

Today, we give a proof of the following:

Theorem 1.4

There exists an existential $\mathcal{L}$-formula φ in 4 free variables such that for all $a, b, c \in F^\times$ one has

$$\bigcap_{v \in \Delta_0\langle\langle a,b,c\rangle\rangle} \mathcal{O}_v = \{x \in F^\times \mid F \models \varphi(x, a, b, c)\}.$$

Plan for today:

- Prove 1.4.

Recall: S-sets of quaternion algebras

Recall that for a field K ($\mathrm{char}(K) \neq 2$) and a quaternion algebra Q we defined

$$S(Q) = \{\mathsf{Trd}(\alpha) \mid \alpha \in Q \setminus K, \mathsf{Nrd}(\alpha) = 1\}$$

Recall: S-sets of quaternion algebras

Recall that for a field K ($\mathrm{char}(K) \neq 2$) and a quaternion algebra Q we defined

$$S(Q) = \{\mathrm{Trd}(\alpha) \mid \alpha \in Q \setminus K, \mathrm{Nrd}(\alpha) = 1\}$$

Proposition 2.1

One has for $a, b \in K^\times$ *with* $Q = (a, b)_K$

$$\begin{aligned} S(Q) &= \{x \in K \mid Q_{K(\sqrt{x^2-4})} \text{ is split}\} \\ &= \{x \in K \mid \langle\langle a, b\rangle\rangle_{K(\sqrt{x^2-4})} \text{ is isotropic}\} \\ &= \{x \in K \mid \langle x^2 - 4, -a, -b, ab\rangle_K \text{ is isotropic}\} \end{aligned}$$

Proof: Exercise. □

Pfister forms and quadratic extensions

Recall: an n-fold Pfister from over a field K $(\mathrm{char}(K) \neq 2)$ is a quadratic form isometric to

$$\langle 1, -a_1 \rangle_K \otimes \cdots \otimes \langle 1, -a_n \rangle_K$$

for certain $a_1, \ldots, a_n \in K^\times$. It is then denoted $\langle\langle a_1, \ldots, a_n \rangle\rangle_K$.

Pfister forms and quadratic extensions

Recall: an n-fold Pfister from over a field K ($\mathrm{char}(K) \neq 2$) is a quadratic form isometric to

$$\langle 1, -a_1 \rangle_K \otimes \cdots \otimes \langle 1, -a_n \rangle_K$$

for certain $a_1, \ldots, a_n \in K^\times$. It is then denoted $\langle\langle a_1, \ldots, a_n \rangle\rangle_K$.

If π is a Pfister from, we can write it as $\langle 1 \rangle \perp \pi'$ for some quadratic form π', called the *pure part* of π.

Pfister forms and quadratic extensions

Recall: an n-fold Pfister from over a field K ($\mathrm{char}(K) \neq 2$) is a quadratic form isometric to

$$\langle 1, -a_1 \rangle_K \otimes \cdots \otimes \langle 1, -a_n \rangle_K$$

for certain $a_1, \ldots, a_n \in K^\times$. It is then denoted $\langle\langle a_1, \ldots, a_n \rangle\rangle_K$.

If π is a Pfister from, we can write it as $\langle 1 \rangle \perp \pi'$ for some quadratic form π', called the *pure part* of π.
Becher mentioned:

Proposition 2.2

Let π be a Pfister form over K, $d \in K^\times$. Then $\pi_{K[\sqrt{d}]}$ is isotropic if and only if $\langle d \rangle \perp \pi'$ is isotropic over K.

We call $\langle d \rangle \perp \pi'$ the *twist of π by d*.

S_d-sets

For a Pfister form q defined over a field K with $\mathrm{char}(K) \neq 2$ and some $d \in K^\times$, we consider the set

$$S_d(q) = \{x \in K \mid q \text{ is isotropic over } K[\sqrt{x^2 + 4d}]\}$$
$$= \{x \in K \mid \langle x^2 + 4d \rangle_K \perp q' \text{ is isotropic}\}.$$

This set has an existential definition in the language of rings, uniformly in d and the parameters defining q.

Basic properties

Proposition 2.3

Let K be a field with $\mathrm{char}(K) \neq 2$, $d \in K^\times$, q a Pfister form defined over K.

1. *If q is isotropic over K, then $S_d(q) = K$.*
2. *If q is anisotropic over K, then $x \in S_d(q)$ implies that $x^2 + 4d$ is a non-square in K.*
3. *If L/K is a field extension, $S_d(q) \subseteq S_d(q_L)$.*

Proof: Clear. □

Basic properties

Proposition 2.4

Let K be a field of $\mathrm{char}(K) \neq 2$, $\mathcal{E}$ a collection of field extensions of K, q a Pfister form over K. Suppose that for all $d \in K^\times$ such that $\langle d \rangle_K \perp q'$ is anisotropic, there exists an $E \in \mathcal{E}$ such that $\langle d \rangle_E \perp q'_E$ is anisotropic. Then

$$S_d(q) = \bigcap_{E \in \mathcal{E}} (S_d(q_E) \cap K).$$

Proof: Exercise. □

Basic properties

Proposition 2.4

Let K be a field of $\mathrm{char}(K) \neq 2$, $\mathcal{E}$ a collection of field extensions of K, q a Pfister form over K. Suppose that for all $d \in K^\times$ such that $\langle d\rangle_K \perp q'$ is anisotropic, there exists an $E \in \mathcal{E}$ such that $\langle d\rangle_E \perp q'_E$ is anisotropic. Then

$$S_d(q) = \bigcap_{E \in \mathcal{E}} (S_d(q_E) \cap K).$$

Proof: Exercise. ☐

Equivalently, if the Pfister form q satisfies a local-global principle with respect to a collection of valuations over K and all quadratic extensions of K, then the equality of S-sets above holds.

Over henselian valued fields

Proposition 2.5

Let $(\mathcal{K}, v)$ be a henselian discretely valued field with $\mathrm{char}(\mathcal{K}) \neq 2$ and q a Pfister form defined over $\mathcal{K}$. Furthermore, let $x \in \mathcal{K}, d \in \mathcal{K}^\times$.

1. *If q is anisotropic over $\mathcal{K}$, then $x \in S_d(q)$ implies $v(x^2) \geq v(d)$.*
2. *If q is isotropic over $K[\sqrt{d}]$, then $v(x^2) > v(16d)$ implies $x \in S_d(q)$*

Proof:

Parametrised solution

Proposition 3.1

Let q be a three-fold Pfister form defined over F. Let $d \in F^\times$ be such that q is isotropic over $F[\sqrt{d}]$. Then

$$K \cdot S_d(q) = \{x \in F \mid \forall w \in \Delta_0 q : w(x^2) \geq w(d)\}.$$

In particular, if $v(d) = 0$ for all $v \in \Delta_0 q$, then

$$K \cdot S_d(q/F) = \bigcap_{w \in \Delta_0 q} \mathcal{O}_w$$

Recall that $S_d(q)$ has an existential definition, uniform in c and the parameters defining q. We saw in the last talk that K is existentially definable in F, hence also $K \cdot S_d(q)$ is existentially definable in F, uniformly in d and the parameters defining q.

Parametrised solution

Lemma 3.2

Let K be a field with $\mathrm{char}(F) \neq 2$, q a quadratic form over K of dimension at least 3, S a finite set of $\mathbb{Z}$-valuations on K. There exists a $d \in K^\times$ such that $q_{K[\sqrt{d}]}$ is isotropic and $v(d) = 0$ for all $v \in S$.

Proof: Exercise. □

Parametrised solution

Lemma 3.2

Let K be a field with $\mathrm{char}(F) \neq 2$, q a quadratic form over K of dimension at least 3, S a finite set of $\mathbb{Z}$-valuations on K. There exists a $d \in K^\times$ such that $q_{K[\sqrt{d}]}$ is isotropic and $v(d) = 0$ for all $v \in S$.

Proof: Exercise. □

Let again F be an algebraic function field over a local field (K, v).

Corollary 3.3

For any three-fold Pfister form q defined over F, the set $\bigcap_{w \in \Delta_0 q} \mathcal{O}_w$ is existentially definable in F.

It follows (exercise) that also each $\mathcal{O}_w$ for $w \in \mathcal{V}_0$ is existentially definable.

Parametrised solution

Proof of Proposition 3.1:

Eliminating the parameter d

- We have shown that there is an existential $\mathcal{L}$-formula φ in 5 free variables such that for all $a, b, c \in F^\times$ and a *good choice* for $d \in F^\times$ we have

$$\bigcap_{v \in \Delta_0\langle\langle a,b,c\rangle\rangle} \mathcal{O}_v = \{x \in F^\times \mid F \models \varphi(x, a, b, c, d)\}.$$

Eliminating the parameter d

- We have shown that there is an existential $\mathcal{L}$-formula φ in 5 free variables such that for all $a, b, c \in F^\times$ and a *good choice* for $d \in F^\times$ we have

$$\bigcap_{v \in \Delta_0\langle\langle a,b,c\rangle\rangle} \mathcal{O}_v = \{x \in F^\times \mid F \models \varphi(x, a, b, c, d)\}.$$

- We would like to get rid of the need to choose an appropriate $d \in F^\times$ (this hinders quantification over the a, b, c when passing to universal formulae).

Eliminating the parameter d

Proposition 3.4

Let q be a three-fold Pfister form defined over F. Then

$$\bigcap_{w \in \Delta_0 q} \mathcal{O}_w = \bigcup_{d \in \mathcal{C}} K \cdot S_d(q)$$

where

$$\mathcal{C} = \left\{ \frac{e}{(e-1)^2} \;\middle|\; e \in F^\times, 0 \in S_e(q) \right\}.$$

Proof:

Eliminating the parameter d

Proof of Theorem 1.4:

Towards universal definitions

With this existential formula for $\bigcap_{w \in \Delta_0 q} \mathcal{O}_w$, we can now give universal definitions of rings of S-integers by using information about 'ramification behaviour' of 3-fold Pfister forms over F.

Towards universal definitions

With this existential formula for $\bigcap_{w\in\Delta_0 q}\mathcal{O}_w$, we can now give universal definitions of rings of S-integers by using information about 'ramification behaviour' of 3-fold Pfister forms over F.

The 'ramification behaviour' we needed is contained in the exact sequence

$$H^3(F)\longrightarrow\bigoplus_{w\in\mathcal{V}_0}H^3(F_w)\cong\bigoplus_{w\in\mathcal{V}_0}\mathbb{Z}/2\longrightarrow\mathbb{Z}/2\longrightarrow 0$$

Towards universal definitions

With this existential formula for $\bigcap_{w \in \Delta_0 q} \mathcal{O}_w$, we can now give universal definitions of rings of S-integers by using information about 'ramification behaviour' of 3-fold Pfister forms over F.

The 'ramification behaviour' we needed is contained in the exact sequence

$$H^3(F) \longrightarrow \bigoplus_{w \in \mathcal{V}_0} H^3(F_w) \cong \bigoplus_{w \in \mathcal{V}_0} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

- Reciprocity Law: the ramification sets are precisely the subsets of $\mathcal{V}_0$ containing an even number of elements

Towards universal definitions

With this existential formula for $\bigcap_{w\in\Delta_0 q}\mathcal{O}_w$, we can now give universal definitions of rings of S-integers by using information about 'ramification behaviour' of 3-fold Pfister forms over F.

The 'ramification behaviour' we needed is contained in the exact sequence

$$H^3(F) \longrightarrow \bigoplus_{w\in\mathcal{V}_0} H^3(F_w) \cong \bigoplus_{w\in\mathcal{V}_0} \mathbb{Z}/2 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$$

- Reciprocity Law: the ramification sets are precisely the subsets of $\mathcal{V}_0$ containing an even number of elements
- $H^3(F)$ (in this case) consists only of symbols ($\leftrightarrow$ Pfister forms)

Outlook: algebraic function fields over $\mathbb{Q}$

Let F be an algebraic function field over a global field K. Let $\mathcal{V}$ be the set of $\mathbb{Z}$-valuations which are trivial on K. Can we still universally define rings of S-integers?

- Work with 3-fold Pfister forms as before.

Outlook: algebraic function fields over $\mathbb{Q}$

Let F be an algebraic function field over a global field K. Let $\mathcal{V}$ be the set of $\mathbb{Z}$-valuations which are trivial on K. Can we still universally define rings of S-integers?

- Work with 3-fold Pfister forms as before.
- There is an existential formula (due to Dittmann and Daans) associating to $(a, b, c) \in (F^\times)^3$ the subring $\bigcap_{v \in \Delta(a,b,c)} \mathcal{O}_v$

Outlook: algebraic function fields over $\mathbb{Q}$

Let F be an algebraic function field over a global field K. Let $\mathcal{V}$ be the set of $\mathbb{Z}$-valuations which are trivial on K. Can we still universally define rings of S-integers?

- Work with 3-fold Pfister forms as before.
- There is an existential formula (due to Dittmann and Daans) associating to $(a, b, c) \in (F^\times)^3$ the subring $\bigcap_{v \in \Delta(a,b,c)} \mathcal{O}_v$
- More subtle ramification behaviour. The complex

$$0 \to H^3(K) \to H^3(F) \to \bigoplus_{v \in \mathcal{V}_0} H^3(F_v) \to H^2(K) \to 0$$

has finite cohomology groups. Elements of $H^3(F)$ might not all be symbols. $H^2(K)$ is much more complicated than $\mathbb{Z}/2$.