Quadratic forms and valued fields

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Talk structure:

1. Quadratic forms (coordinate-freely)
2. Valued fields & quadratic forms: first observations
3. Residue forms
4. Applications

Section 1

Quadratic forms (coordinate-freely)

Quadratic forms

Let K be a field.

Definition 1.1

A *quadratic space over* K is a pair (V, Q) where V is a finite-dimensional K-vector space, and Q is a map $V \to K$ such that

- $\forall a \in K, v \in V : Q(av) = a^2 Q(v)$, and
- the map

$$\mathfrak{b}_Q : V \times V \to K : (x, y) \mapsto Q(x+y) - Q(x) - Q(y)$$

  is bilinear.

We call the map Q a *quadratic form over* K, and the map $\mathfrak{b}_Q$ the *associated bilinear form*.

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The *dimension of* Q is by definition $\dim(V)$; we may write this as $\dim(Q)$ as well.

Quadratic forms

Note: Often, an n-dimensional quadratic form over K is defined to be a homogeneous degree 2 polynomial $H \in K[X_1, \ldots, X_n]$. For example:

- $H(X_1, X_2) = X_1 X_2$,
- $H(X_1, X_2, X_3) = X_1^2 + X_2^2 - X_3^2$,
- $H(X_1, X_2, X_3, X_4) = (X_1 + X_2)^2 - 4(X_3 - X_4)^2$.

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Our definition is simply a coordinate-free reformulation of this.

Quadratic forms

Proposition 1.2

Let (V, Q) be a quadratic space over K, let $(b_1, \ldots, b_n)$ be a K-basis of V.
There exists a homogeneous degree 2 polynomial $H \in K[X_1, \ldots, X_n]$ such that for all $a_1, \ldots, a_n \in K$ we have

$$Q(a_1 b_1 + \ldots + a_n b_n) = H(a_1, \ldots, a_n).$$

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$$Q(a_1 b_1 + \dots + a_n b_n) = H(a_1, \dots, a_n).$$

Conversely, given a homogeneous degree 2 polynomial $H \in K[X_1, \dots, X_n]$, the map

$$K^n \to K : (a_1, \dots, a_n) \mapsto H(a_1, \dots, a_n)$$

is a quadratic form on K^n.

Proof: Exercise.

Quadratic forms

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We call two quadratic spaces (V_1, Q_1) and (V_2, Q_2) *isometric* (denoted by $(V_1, Q_1) \cong (V_2, Q_2)$ or simply $Q_1 \cong Q_2$) if there exists an isomorphism of K-vector spaces $\iota : V_1 \to V_2$ such that $Q_2(\iota(v)) = Q_1(v)$ for all $v \in V_1$.

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For quadratic spaces (V_1, Q_1) and (V_2, Q_2), we define its orthogonal sum $(V_1, Q_1) \perp (V_2, Q_2)$ as the space $V_1 \times V_2$ with the form

$$Q_1 \perp Q_2 : V_1 \times V_2 \to K : (v_1, v_2) \mapsto Q(v_1) + Q(v_2).$$

Quadratic forms

If (V, Q) is a quadratic space over K and L/K a field extension, we denote by (V^L, Q^L) the *extension of* (V, Q) *to* L, i.e. $V^L = V \otimes_K L$ and Q^L is such that $Q^L(x \otimes a) = a^2 Q(x)$ for $x \in V$ and $a \in L$.
We say that Q is *isotropic (resp. anisotropic) over* L if Q^L is isotropic (resp. anisotropic).

Section 2

Valued fields & quadratic forms: first observations

Henselian valued fields

Let K be a field. Consider a valuation v on K, i.e. a group homomorphism

$$v : K^\times \to \Gamma$$

where Γ is an ordered abelian group (written additively) and such that $v(x+y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K$. (We take the convention that $v(0) = \infty > \gamma$ for all $\gamma \in \Gamma$). We call the pair (K, v) a *valued field*.

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- *valuation ring* $\mathcal{O}_v = \{x \in K \mid v(x) \geq 0\}$
- *valuation ideal* $\mathfrak{m}_v = \{x \in K \mid v(x) > 0\}$
- *residue field* $Kv = \mathcal{O}_v/\mathfrak{m}_v$
- *value group* $vK = v(K^\times)$ (if $vK = \mathbb{Z}$, then v is a *$\mathbb{Z}$-valuation*)

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For $a \in \mathcal{O}_v$, we denote by $\overline{a}^v$ (or simply $\overline{a}$) its equivalence class in $\mathcal{O}_v/\mathfrak{m}_v$. Similarly, for $f \in \mathcal{O}_v[X_1, \ldots, X_n]$, denote by $\overline{f}^v$ (or $\overline{f}$) the corresponding element of $Kv[X_1, \ldots, X_n]$.

Henselian valued fields

A valuation v on K is called *henselian* if one of the following equivalent properties holds (see also [EP05, Theorem 4.1.3]):

1. For every finite field extension L/K, there is a unique extension of v to L,
2. *(Hensel's Lemma)* For each $f \in \mathcal{O}_v[X]$ and $a \in \mathcal{O}_v$ with $\overline{f(a)} = 0 \neq \overline{f'(a)}$, there exists an $\alpha \in \mathcal{O}_v$ with $f(\alpha) = 0$ and $\overline{\alpha} = \overline{a}$.

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Examples:

- For a prime p, $(\mathbb{Q}_p, v_p)$ is a henselian $\mathbb{Z}$-valued field with $\mathbb{Q}_p v_p \cong \mathbb{F}_p$.
- For a field K, $(K((t)), v_t)$ is a henselian $\mathbb{Z}$-valued field with $K((t))v_t \cong K$.

Henselian valued fields

If (K, v) is a valued field, then one of two things happens:

- $\mathrm{char}(K) = \mathrm{char}(Kv)$,
- $\mathrm{char}(K) = 0$ and $\mathrm{char}(Kv) \neq 0$.

In either case, for a prime number p, we have $\mathrm{char}(Kv) = p$ if and only if $p \in \mathfrak{m}_v$, i.e. $v(p) > 0$.

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In the context of quadratic forms: we call a valued field (K, v) *dyadic* if $\mathrm{char}(Kv) = 2$ (equivalently, $v(2) > 0$), *non-dyadic* otherwise.

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E.g. $\mathbb{Q}_2$ and $\mathbb{F}_2((t))$ are dyadic, $\mathbb{Q}_3$ and $\mathbb{F}_3((t))$ are non-dyadic.

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Goal: Understand quadratic forms over valued fields (classify, study properties).

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(**Spoiler:** Yes if v is non-dyadic. Otherwise, it's complicated.)

Quadratic forms over $\mathcal{O}_v$ and their residues

Let (K, v) be a valued field.

Definition 2.1 (Quadratic form over valuation ring)

A *quadratic space over* $\mathcal{O}_v$ is a pair (V, Q) where V is a free finite-dimensional $\mathcal{O}_v$-module and $Q : V \to \mathcal{O}_v$ is a map such that

- $\forall a \in K, v \in V : Q(av) = a^2 Q(v)$, and
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As before, after giving coordinates, quadratic forms over $\mathcal{O}_v$ correspond to homogeneous degree 2 polynomials over $\mathcal{O}_v$.

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- On the polynomial level, given a homogeneous degree 2 polynomial $H \in \mathcal{O}_v[X_1, \ldots, X_n]$, extend scalars to K by simply considering it as a polynomial over K, and take residues by considering $\overline{H}^v \in Kv[X_1, \ldots, X_n]$.

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- Note: a quadratic form space (V, Q) is isotropic if and only if (V^K, Q^K) is isotropic.

Quadratic forms over $\mathcal{O}_v$ and their residues

Fix a valued field (K, v).

Proposition 2.2

Let (V, Q) be a quadratic space over $\mathcal{O}_v$. Assume that $\overline{Q}^V$ is anisotropic. Then for $x \in V \setminus \mathfrak{m}_v V$ one has $v(Q(x)) = 0$. In particular, Q is anisotropic and represents only elements of even value.

Proof: Exercise.

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On the polynomial level, this says: if $H \in \mathcal{O}_v[X_1, \ldots, X_n]$ is such that $\overline{H}^v$ is anisotropic, then H is anisotropic, and in fact

$$v(H(a_1, \ldots, a_n)) = 2 \min\{v(a_1), \ldots, v(a_n)\}$$

for all $a_1, \ldots, a_n \in K$.

Quadratic forms over $\mathcal{O}_v$ and their residues

Corollary 2.3

Let $m \in \mathbb{N}$. Suppose that $\pi_1, \ldots, \pi_m \in K^\times$ are such that $v(\pi_1), \ldots, v(\pi_m)$ represent different classes in $vK/2vK$. Let $(V_1, Q_1), \ldots, (V_m, Q_m)$ be quadratic spaces over $\mathcal{O}_v$ such that $\overline{Q_i}^v$ is anisotropic for each i. Then the quadratic space

$$(V_1^K \times \ldots \times V_m^K,\ \pi_1 Q_1^K \perp \ldots \perp \pi_m Q_m^K)$$

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Proof: Let $(w_1, \ldots, w_m) \in (V_1^K \times \ldots \times V_m^K) \setminus \{0\}$. Let $i \in \{1, \ldots, m\}$ be such that $w_i \neq 0$.

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Quadratic forms over $\mathcal{O}_v$ and their residues

To formulate a partial converse, we need the following definitions.

Definition 2.4

Let (V, Q) be a quadratic space over K. A *nonsingular zero of Q* is a vector $v \in V$ such that $Q(v) = 0$, but the map

$$\mathfrak{b}_Q(v, \cdot) : V \to K : w \mapsto \mathfrak{b}_Q(v, w)$$

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Equivalently, a nonsingular zero of Q is a smooth point on the projective variety defined by Q.

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Proposition 2.5

Assume that (K, v) is henselian. Let (V, Q) be a quadratic space over $\mathcal{O}_v$. If $\overline{Q}^v$ has a nonsingular zero, then so does Q.

Quadratic forms over $\mathcal{O}_v$ and their residues

In summary, for a quadratic space (V, Q) over a henselian valuation ring $\mathcal{O}_v$:

- If Q is isotropic, then $\overline{Q}^v$ is isotropic.
- If $\overline{Q}^v$ has a nonsingular zero, then Q has a nonsingular zero (hence in particular is isotropic).

Note: If $\overline{Q}^v$ is isotropic but has no nonsingular zeroes, then we cannot conclude anything about Q.

Examples

Consider over the field of 3-adic numbers $\mathbb{Q}_3$ the form

$$H(X_1, X_2, X_3) = X_1^2 + 2X_1X_2 + 4X_2^2 + X_3^2.$$

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Taking residues, we obtain over $\mathbb{F}_3$ the form

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It has a singular zero $(1, -1, 0)$, but no nonsingular zeroes. Hence, we canot conclude right away.

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Let us do a variable transformation (= isometry).

$$H(X_1 - X_2, X_2, X_3) = \underbrace{X_1^2 + X_3^2}_{Q_1(X_1,X_3)} + 3 \underbrace{X_2^2}_{Q_2(X_2)} .$$

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It has a singular zero $(1, -1, 0)$, but no nonsingular zeroes. Hence, we canot conclude right away.
Let us do a variable transformation (= isometry).

$$H(X_1 - X_2, X_2, X_3) = \underbrace{X_1^2 + X_3^2}_{Q_1(X_1,X_3)} + 3 \underbrace{X_2^2}_{Q_2(X_2)} .$$

Since $\overline{Q_1}$ and $\overline{Q_2}$ are anisotropic, and $v_3(3) = 1 \notin 2v_3\mathbb{Q}_3$, we conclude that H is anisotropic.

Examples

Consider over the field of 2-adic numbers $\mathbb{Q}_2$ the form

$$H(X_1, X_2, X_3, X_4) = X_1^2 + X_2^2 + X_3^2 + X_4^2.$$

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It has a singular zero $(1, 1, 1, 1)$, but no nonsingular zeroes. Hence, we cannot conclude right away.

Examples

Let us do a variable transformation

$$H\left(\frac{X_1}{2}, \frac{X_1}{2} + X_2 + X_3, \frac{X_1}{2} + X_3 + X_4, \frac{X_1}{2} + X_4\right)$$

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\begin{aligned}
&H\left(\frac{X_1}{2}, \frac{X_1}{2}+X_2+X_3, \frac{X_1}{2}+X_3+X_4, \frac{X_1}{2}+X_4\right)\\
= \ &\underbrace{X_1^2+X_1X_2+X_2^2}_{(\mathcal{O}_v^2, Q_1)} + 2\underbrace{\left(X_3^2+X_3X_4+X_4^2\right)}_{(\mathcal{O}_v^2, Q_2)}\\
&+2\underbrace{\left(X_1X_3+X_1X_4+X_2X_3+X_3X_4\right)}_{(\mathcal{O}_v^4, Q_3)}.
\end{aligned}
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Examples

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$$\begin{aligned} & H\left(\frac{X_1}{2}, \frac{X_1}{2}+X_2+X_3, \frac{X_1}{2}+X_3+X_4, \frac{X_1}{2}+X_4\right) \\ = & \underbrace{X_1^2+X_1X_2+X_2^2}_{(\mathcal{O}_v^2, Q_1)} + 2\underbrace{\left(X_3^2+X_3X_4+X_4^2\right)}_{(\mathcal{O}_v^2, Q_2)} \\ & + 2\underbrace{\left(X_1X_3+X_1X_4+X_2X_3+X_3X_4\right)}_{(\mathcal{O}_v^4, Q_3)}. \end{aligned}$$

We compute that $\overline{Q_1}^v$ and $\overline{Q_2}^v$ are anisotropic and that, for any $w_1, w_2 \in \mathcal{O}_v^2$, one has $2v(Q_3(w_1, w_2)) \geq v(Q_1(w_1)) + v(Q_2(w_2))$.

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How to formulate these techniques in a general framework, which does not depend on a choice of coordinates?

Section 3

Residue forms

The Schwarz Inequality and residue forms

Proposition 3.1 (Schwarz Inequality)

Let (K, v) be a henselian valued field. Let (V, Q) be a quadratic space over K. If there exist $w_1, w_2 \in V$ such that

$$2v(\mathfrak{b}_Q(w_1, w_2)) < v(Q(w_1)) + v(Q(w_2))$$

then Q has a nonsingular zero.

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Proof: After rescaling w_1 and w_2, we may assume that $v(\mathfrak{b}_Q(w_1, w_2)) = 0 \leq v(Q(w_1)) \leq v(Q(w_2))$ and $v(Q(w_2)) > 0$.

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$$f(T) = Q(w_2) + T\mathfrak{b}_Q(w_1, w_2) + T^2 Q(w_1) \in \mathcal{O}_v[T].$$

Since $\overline{f(0)} = \overline{Q(w_2)} = 0$ but $\overline{f'(0)} = \overline{\mathfrak{b}_Q(w_1, w_2)} \neq 0$, f has a root $\alpha \in \mathcal{O}_v$ since (K, v) is henselian.

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Proof continued: We compute that

$$Q(w_2 + \alpha w_1) = Q(w_2) + \alpha \mathfrak{b}_Q(w_1, w_2) + \alpha^2 Q(w_1) = 0, \text{ and}$$
$$\mathfrak{b}_Q(w_2 + \alpha w_1, w_1) = \underbrace{\mathfrak{b}_Q(w_2, w_1)}_{\in \mathcal{O}_v^\times} + \underbrace{\alpha Q(w_2)}_{\in \mathfrak{m}_v} \neq 0$$

so $w_2 + \alpha w_1$ is a nonsingular zero of Q. $\square$

The Schwarz Inequality and residue forms

Now let (K, v) be a henselian valued field and (V, Q) an *anisotropic* quadratic space over V.

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$$V_\gamma = \{w \in V \mid v(Q(w)) \geq \gamma\} \text{ and}$$
$$V_\gamma^+ = \{w \in V \mid v(Q(w)) > \gamma\}$$

are $\mathcal{O}_v$-submodules of V, and V_γ / V_γ^+ naturally becomes a Kv-vector space.

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We call these forms $\mathrm{r}_\pi(Q)$ (for different choices of π) *residue forms of Q (with respect to v)*.

The Schwarz Inequality and residue forms

Example: consider as before over $\mathbb{Q}_3$ the form

$$H(X_1, X_2, X_3) = X_1^2 + 2X_1X_2 + 4X_2^2 + X_3^2,$$

seen as a quadratic form on $\mathbb{Q}_3^3$.

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From this, one can compute (exercise) that

$$\begin{aligned} V_0 &= \mathcal{O}_v^3, \\ V_1 &= V_0^+ = \{(a, b, c) \in \mathcal{O}_v^3 \mid a + b, c \in \mathfrak{m}_v\}, \\ V_2 &= V_1^+ = \mathfrak{m}_v V_0 = \mathfrak{m}_v^3. \end{aligned}$$

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We see that $V_0/V_1 = \langle(1, 0, 0), (0, 0, 1)\rangle$ is 2-dimensional and $V_1/V_2 = \langle(1, -1, 0)\rangle$ is 1-dimensional.

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We see that $V_0/V_1 = \langle (1,0,0), (0,0,1) \rangle$ is 2-dimensional and $V_1/V_2 = \langle (1,-1,0) \rangle$ is 1-dimensional. We can now compute the residue forms:

$$\begin{aligned} \mathrm{r}_1(H) &\cong \overline{H(X_1(1,0,0) + X_2(0,0,1))} = X_1^2 + X_2^2 \\ \mathrm{r}_3(H) &\cong \overline{3^{-1}H(X_1(1,-1,0))} = X_1^2 \end{aligned}$$

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$\Rightarrow$ Up to rescaling, one can associate residue forms of Q to elements of $vK/2vK$.
2. Let $m \in \mathbb{N}$. Suppose that $\pi_1, \ldots, \pi_m \in K^\times$ are such that $v(\pi_1), \ldots, v(\pi_m)$ represent different classes in $vK/2vK$. Let $(V_1, Q_1), \ldots, (V_m, Q_m)$ be quadratic spaces over $\mathcal{O}_v$ such that $\overline{Q_i}^v$ is anisotropic for each i. Then the anisotropic quadratic space

$$(V, Q) = (V_1^K \times \ldots \times V_m^K,\ \pi_1 Q_1^K \perp \ldots \perp \pi_m Q_m^K)$$

has residue forms $\mathrm{r}_{\pi_i}(Q) \cong \overline{Q_i}^v$ for $i = 1, \ldots, m$, and $\mathrm{r}_\pi(Q) = 0$ for $\pi \in K^\times$ with $v(\pi) \notin v(\pi_i) + 2vK$ for all i.

Non-dyadic henselian valued fields

Theorem 3.2

Let (K, v) be a non-dyadic henselian valued field.

1. *Every regular quadratic space over K is isometric to a space of the form $(V_1^K \times \ldots \times V_m^K,\ \pi_1 Q_1^K \perp \ldots \perp \pi_m Q_m^K)$ where $\pi_1, \ldots, \pi_m \in K$ are such that $v(\pi_1), \ldots, v(\pi_m)$ represent different elements in $vK/2vK$, (V_i, Q_i) is a quadratic space over $\mathcal{O}_v$, and $\overline{Q_i}$ is regular.*

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2. *Given anisotropic quadratic spaces (V_1, Q_1) and (V_2, Q_2) over K such that $\mathrm{r}_\pi(Q_1) \cong \mathrm{r}_\pi(Q_2)$ for all $\pi \in K^\times$, we have $Q_1 \cong Q_2$.*

Subtleties in the dyadic case

Let (K, v) be a dyadic henselian valued field. Let $(\pi_\gamma)_{\gamma \in vK/2vK} \in K^{vK/2vK}$ be such that $v(\pi_\gamma) \in \gamma$.

Subtleties in the dyadic case

Let (K, v) be a dyadic henselian valued field. Let $(\pi_\gamma)_{\gamma \in vK/2vK} \in K^{vK/2vK}$ be such that $v(\pi_\gamma) \in \gamma$.

- For an anisotropic quadratic space (V, Q), we have $\sum_{\gamma \in vK/2vK} \dim(\mathrm{r}_{\pi_\gamma}(Q)) \leq \dim(Q)$, but we might not have equality.

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- Even if we do have $\sum_{\gamma \in vK/2vK} \dim(\mathrm{r}_{\pi_\gamma}(Q)) = \dim(Q)$, the forms $\mathrm{r}_{\pi_\gamma}(Q)$ do not in general uniquely determine the quadratic space (V, Q) up to isometry.

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- There could be an anisotropic quadratic space (V, Q) and a valued field extension (K', v') such that Q is isotropic over K', but $vK = v'K'$ and $Kv = K'v'$.

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Let (K, v) be a dyadic henselian valued field. Let $(\pi_\gamma)_{\gamma \in vK/2vK} \in K^{vK/2vK}$ be such that $v(\pi_\gamma) \in \gamma$.

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- There could be an anisotropic quadratic space (V, Q) and a valued field extension (K', v') such that Q is isotropic over K', but $vK = v'K'$ and $Kv = K'v'$.

See the exercises for examples.

Subtleties in the dyadic case

Theorem 3.3 (Mammone, Moresi, Wadsworth [MMW91])

Let (K, v) a henselian $\mathbb{Z}$-valued field with uniformiser π, let Q be a quadratic form over K. Then $\dim(Q) = \dim(\mathrm{r}_1(Q)) + \dim(\mathrm{r}_\pi(Q))$ as soon as one of the following holds:

- $\mathrm{char}(K) \neq 2$,
- *Q is non-degenerate (= regular over the algebraic closure $\overline{K}$),*
- $\mathrm{char}(K) = 2$ *and* $[K : K^2] = 2[Kv : Kv^2]$.

Subtleties in the dyadic case

Definition 3.4

Let K be a field. A quadratic space (V, Q) is called *nonsingular* if the associated bilinear form $\mathfrak{b}_Q$ is nonsingular, i.e.

$$\forall v \in K \ \exists w \in K : Q(v+w) \neq Q(v) + Q(w).$$

Nonsingular quadratic spaces are always regular. Over fields of characteristic different from two, the converse holds.

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$$\forall v \in K \ \ \exists w \in K : Q(v+w) \neq Q(v) + Q(w).$$

Nonsingular quadratic spaces are always regular. Over fields of characteristic different from two, the converse holds.

Examples:

- $X_1^2 + X_1X_2 + X_2^2$ is nonsingular over $\mathbb{F}_2$,
- $X_1^2 + X_1X_2 + X_2^2 + X_3^2$ is regular but not nonsingular over $\mathbb{F}_2$.

Theorem 3.5 (Tietze; Elomary, Tignol [Tie74; ET11])

Let (K, v) be a dyadic henselian valued field, Q an anisotropic quadratic form over K. The following are equivalent:

1. *Q is hyperbolic over some inert extension (L, w) of (K, v) (i.e. $(L, w)/(K, v)$ is separable algebraic and defectless, Lw/Kv is separable, and $vK = wL$),*
2. *all residue forms of Q are nonsingular,*
3. *Q is isometric to an orthogonal sum of forms of the form $a_1X^2 + a_2XY + a_3Y^2$ with $v(a_2) \leq \frac{v(a_1)+v(a_3)}{2}$.*

When Q satisfies these equivalent conditions, then $\dim(Q) = \sum_{\gamma \in vK/2vK} \mathrm{r}_{\pi_\gamma}(Q)$, and if Q' is another anisotropic quadratic form with $\mathrm{r}_\pi(Q) \cong \mathrm{r}_\pi(Q')$ for all $\pi \in K^\times$, then $Q \cong Q'$.

Section 4

Applications

Classification of quadratic forms

Residue forms can be used to study the quadratic form theory of a henselian valued field (K, v) via that of its residue field. See exercises.

Classification of quadratic forms

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Proposition 4.1

Let (K, v) be a henselian $\mathbb{Z}$-valued field, $n \in \mathbb{N}$. Suppose that every $(n+1)$-dimensional quadratic form over Kv is isotropic. Then every nondegenerate $(2n+1)$-dimensional quadratic form over K is isotropic.

Proof: Exercise.

Classification of quadratic forms

Proposition 4.2

Let (K, v) be a non-dyadic henselian valued field, $n \in \mathbb{N}$. Suppose that every $(n+1)$-dimensional quadratic form over Kv is isotropic, and there is a unique anisotropic n-dimensional quadratic form over Kv up to isometry. Then every $([vK : 2vK]n + 1)$-dimensional quadratic form over K is isotropic, and there is a unique anisotropic $[vK : 2vK]n$-dimensional quadratic form over K.

Proof: Exercise.

Quadratic forms under field extensions

Proposition 4.3

Let (K, v) be a henselian valued field, $(L, w)/(K, v)$ a valued field extension such that $wL = vK$. Let $\pi_1, \ldots, \pi_m \in K^\times$ such that $v(\pi_1), \ldots, v(\pi_m)$ represent different classes in $vK/2vK$. Assume that (V, Q) is an anisotropic quadratic space over K with $\dim(Q) = \sum_{i=1}^{m} \mathrm{r}_{\pi_i}(Q)$. Then

Quadratic forms under field extensions

Proposition 4.3

Let (K, v) be a henselian valued field, $(L, w)/(K, v)$ a valued field extension such that $wL = vK$. Let $\pi_1, \ldots, \pi_m \in K^\times$ such that $v(\pi_1), \ldots, v(\pi_m)$ represent different classes in $vK/2vK$. Assume that (V, Q) is an anisotropic quadratic space over K with $\dim(Q) = \sum_{i=1}^m \mathrm{r}_{\pi_i}(Q)$. *Then*

1. *if $\mathrm{r}_{\pi_i}(Q)$ is anisotropic over Lw for all i, then Q^L is anisotropic,*

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Proof sketch of (1): Let $\gamma_i = v(\pi_i)$. Let $\mathcal{B}_i \subseteq V_{\gamma_i}$ be a linearly independent set such that $\{\overline{x} \mid x \in \mathcal{B}_i\}$ is a basis for $V_{\gamma_i}/V_{\gamma_i}^+$. Let $W_i = \langle \mathcal{B}_i \rangle_{\mathcal{O}_v}$. For $x \in W_i^L \setminus \{0\}$ we have $v(q(x)) \in \gamma_i + 2wL$. Since (by comparing dimensions) $V^L = W_1^L \oplus \ldots \oplus W_m^L$, and by the Schwarz Inequality, we obtain that Q^L is anisotropic. $\square$

References

[EP05] Antonio J. Engler and Alexander Prestel. *Valued Fields*. Springer, 2005.

[ET11] Mohamed Abdou Elomary and Jean-Pierre Tignol. “Springer’s theorem for tame quadratic forms over Henselian fields”. In: *Mathematische Zeitschrift* 269 (2011), pp. 309–323.

[MMW91] Pasquale Mammone, Remo Moresi, and Adrian R. Wadsworth. “u-invariants of fields of characteristic 2.”. In: *Mathematische Zeitschrift* 208.3 (1991), pp. 335–348.

[Tie74] Uwe-Peter Tietze. “Zur Theorie quadratische Formen über Hensel-Körpern”. In: *Archiv der Mathematik* 25 (1974), pp. 144–150.

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