University of Antwerp
Faculty of Science

Model-theoretic tools for the busy algebraist

Antwerp Algebra Colloquium

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Structures and signatures

General framework to study classes of mathematical (algebraic) objects defined by "constants, operations and relations"

- A ring is a set A endowed with binary operations $+, -, \cdot : A^2 \to A$ and two constants 0 and 1 (+ some conditions)
- A poset is a set A endowed with a binary relation $\leq$, i.e. a subset of A^2 (+ some conditions)
- An ordered group is a set A endowed with binary operations $+, - : A^2 \to A$, a constant 0, and one binary relation $\leq$ (+ some conditions)
- An $\mathbb{R}$-vector space is a set A endowed with binary operations $+, - : A^2 \to A$, a constant 0, and for each $r \in \mathbb{R}$, a unary operation $r\cdot : A \to A$ (+ some conditions)

Structures and signatures

Formally: we fix a *signature* (or *language*) $\mathcal{L}$, this is a tuple $((I_n)_{n\in\mathbb{N}^+}, (J_n)_{n\in\mathbb{N}^+}, K)$ where I_n, J_n and K are sets. Elements of I_n are called *n-ary function symbols*, elements of J_n *n-ary relation symbols* and elements of K *constant symbols*.

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- The *signature of rings* $\mathcal{L}_{\text{ring}}$: $K = \{0, 1\}$, $I_2 = \{+, -, \cdot\}$, all other I_n and J_n are empty.
- The *signature of* $\mathbb{R}$*-vector spaces* $\mathcal{L}_{\mathbb{R}\,\text{vec}}$: $K = \{0\}$, $I_1 = \mathbb{R}$, $I_2 = \{+, -\}$, all other I_n and J_n are empty.

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Slightly less formally (abuse of notation): we write a signature as a set of symbols of which it is clear from the context whether they are constant symbols, function symbols of some fixed arity, or relation symbols of a fixed arity.

- The *signature of ordered groups* $\mathcal{L}_{\text{ordgrp}} = \{+, -, 0, \leq\}$.

Structures and signatures

Given a signature $\mathcal{L} = ((I_n)_{n\in\mathbb{N}^+}, (J_n)_{n\in\mathbb{N}^+}, K)$, an *$\mathcal{L}$-structure* is a set A endowed with

- for each $c \in K$, a fixed element $c^A \in A$,
- for each $f \in I_n$, a map $f^A : A^n \to A$,
- for each $R \in J_n$, a set $R^A \subseteq A^n$.

In this way, rings naturally carry an $\mathcal{L}_{\text{ring}}$-structure, $\mathbb{R}$-vector spaces naturally carry an $\mathcal{L}_{\mathbb{R}\,\text{vec}}$-structure, etc.

Morphisms, substructures, . . .

A *homomorphism* between $\mathcal{L}$-structures A and B is a map $H : A \to B$ compatible with constants, functions, relations in $\mathcal{L}$. An *isomorphism* between $\mathcal{L}$-structures is a bijective homomorphism whose inverse is a homomorphism.

Similarly one defines *embeddings* of $\mathcal{L}$-structures and $\mathcal{L}$*-substructures*. One can define what it means for an $\mathcal{L}$-structure to be *finitely generated*, *finitely presented*, etc.

A few caveats

- Constants, functions and relations of an $\mathcal{L}$-structure need not satisfy any properties. In particular, an $\mathcal{L}_{\mathsf{ordgrp}}$-structure in general does not correspond to an ordered group (associativity, invertibility, transitivity, etc. need not hold), an $\mathcal{L}_{\mathbb{R}\,\mathsf{vec}}$-structure need not in general correspond to an $\mathbb{R}$-vector space, ...

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- Why did we choose $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$ and not $\{+, \cdot\}$, or $\{+, \cdot, 0\}$, or $\{+, -, \cdot, ^2, 0, 1\}$, ...?

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  - To some extent, the choice is arbitrary, choose what is useful in the context.
  - Rule of thumb: make a minimal choice such that the notions of $\mathcal{L}$-homomorphism, $\mathcal{L}$-substructure, etc. correspond to the natural notion (in this case group homomorphism, subgroup, ...)

$\mathcal{L}$-terms and $\mathcal{L}$-sentences

Constant symbols and function symbols can be formally combined with *variables* (or *variable symbols*) into *$\mathcal{L}$-terms*.

- e.g. in $\mathcal{L}_{\text{ring}}$, a term could be $(x_1 + 1) \cdot (x_2 \cdot x_2) - x_3$; its variables are (x_1, x_2, x_3).

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- If t is an $\mathcal{L}$-term in the variables $(x_1, \ldots, x_n)$, A is an $\mathcal{L}$-structure and $a_1, \ldots, a_n \in A$, we obtain an element $t^A(a_1, \ldots, a_n) \in A$ by "plugging in" $a_1, \ldots, a_n$ for $x_1, \ldots, x_n$.

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- Homomorphisms of $\mathcal{L}$-structures are compatible with evaluation of $\mathcal{L}$-terms.

$\mathcal{L}$-terms and $\mathcal{L}$-sentences

$\mathcal{L}$-terms can be used to make $\mathcal{L}$-*sentences* by meaningfully combining them with relation symbols of $\mathcal{L}$ and the logical symbols $\forall, \exists, \vee, \wedge, \neg, \doteq$ (and delimeters like ()).

- Examples of $\mathcal{L}_{\text{ring}}$-sentences:

$$\forall x \forall y (x + y \doteq y + x)$$
$$\forall x (x \doteq 0 \vee \exists y (x \cdot y \doteq 1))$$
$$\forall y \exists x (x \cdot x \doteq y)$$
$$\exists x_1 \exists x_2 \forall y (y \doteq x_1 \vee y \doteq x_2)$$

- A set of $\mathcal{L}$-sentences is called an $\mathcal{L}$-*theory*.

$\mathcal{L}$-terms and $\mathcal{L}$-sentences

An $\mathcal{L}$-sentence φ can *hold* in an $\mathcal{L}$-structure A, which we denote by $A \models \varphi$. If Σ is an $\mathcal{L}$-theory, we say that Σ *holds in* A, or that A *is a model of* Σ if $A \models \varphi$ for all $\varphi \in \Sigma$; we might denote this by $A \models \Sigma$.

- An ordered group is an $\mathcal{L}_{\text{ordgrp}}$-structure which is a model of the theory consisting of

$$
\begin{aligned}
&\forall x \forall y (x + y \doteq y + x) \\
&\forall x \forall y \forall z (x + y \doteq z \vee \neg(x \doteq z - y)) \\
&\forall x \forall y \forall z ((x + y) + z \doteq x + (y + z)) \\
&\forall x (x + 0 \doteq x) \\
&\forall x \forall y (x \leq y \vee y \leq x) \\
&\forall x \forall y (x \doteq y \vee \neg(x \leq y) \vee \neg(y \leq x)) \\
&\forall x \forall y \forall z (x + z \leq y + z \vee \neg(x \leq y)) \\
&\dots
\end{aligned}
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- In $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$, any two algebraically closed fields of the same characteristic are elementarily equivalent.

The Löwenheim-Skolem Theorem

Given a signature $\mathcal{L} = ((I_n)_{n\in\mathbb{N}^+}, (J_n)_{n\in\mathbb{N}^+}, K)$, its *cardinality* is defined to be

$$\sum_{n\in\mathbb{N}} |I_n| + \sum_{n\in\mathbb{N}} |J_n| + |K| + \aleph_0$$

and denoted $|\mathcal{L}|$. Note that $|\mathcal{L}|$ is the cardinality of the set of $\mathcal{L}$-formulas.

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Theorem (Löwenheim-Skolem)

Let $\mathcal{L}$ be a signature, λ a cardinal with $\lambda \geq |\mathcal{L}|$, A an infinite $\mathcal{L}$-structure. There exists an $\mathcal{L}$-structure B with $A \equiv B$ and $|B| = \lambda$.

Note: the actual Löwenheim-Skolem Theorem says more about how A and B can be chosen to relate.

The Löwenheim-Skolem Theorem

Corollary

Let $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$. Any two algebraically closed fields of the same characteristic are elementarily equivalent as $\mathcal{L}_{\text{ring}}$-structures.

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Corollary

Let $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$*. Any two algebraically closed fields of the same characteristic are elementarily equivalent as* $\mathcal{L}_{\text{ring}}$*-structures.*

Proof.

- Suppose K and L are algebraically closed fields of the same characteristic and with $|K| = |L| = \lambda > \aleph_0$. **Claim:** $K \cong L$.

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- Suppose that K and L are algebraically closed fields of the same characteristic. By the Löwenheim-Skolem Theorem, there exist fields K' and L' such that $K \equiv K'$, $L \equiv L'$ and $|K'| = |L'| = \aleph_1$. By the first part, $K' \cong L'$. We conclude that $K \equiv K' \equiv L' \equiv L$.

□

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- Example: let $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$, φ an $\mathcal{L}_{\text{ring}}$-sentence such that $K \models \varphi$ whenever K is a field with char $K = 0$. Then $\exists m \in \mathbb{N}$ such that $K \models \varphi$ whenever K is a field with char $K \geq m$.

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- Example: let $\mathcal{L}_{\text{ring}} = \{+, -, \cdot, 0, 1\}$, φ an $\mathcal{L}_{\text{ring}}$-sentence such that $K \models \varphi$ whenever K is a field with $\operatorname{char} K = 0$. Then $\exists m \in \mathbb{N}$ such that $K \models \varphi$ whenever K is a field with $\operatorname{char} K \geq m$.
- The Löwenheim-Skolem Theorem and the Compactness Theorem require (some form of) the Axiom of Choice to prove.

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The Ax-Grothendieck Theorem

Theorem (Ax-Grothendieck)

Let $n \in \mathbb{N}$, $V \subseteq \mathbb{C}^n$ an algebraic set over $\mathbb{C}$, $f : V \to V$ an injective rational map. Then f is surjective.

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Consider for a field K the following property

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- Finite fields satisfy (1) by the Pigeon Hole Principle.
- Algebraic extensions of finite fields satisfy (1): an algebraic set V, a polynomial map $f : V \to V$ and an element $x \in V$ are all defined over a finite subfield, so this reduces to the previous case.
  In particular, there exist algebraically closed fields of each positive characteristic p satisfying (1).

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- Property (1) is described by an $\mathcal{L}_{\text{ring}}$-theory Σ, i.e. a field satisfies (1) if and only if it is a model of Σ.

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- By the Compactness Theorem, there exists an algebraically closed field K of characteristic 0 satisfying (1).
- Since $K \equiv \mathbb{C}$, also $\mathbb{C}$ satisfies (1).

□

Further reading

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