Defining valuation and holomorphy rings in function fields using quadratic forms

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(joint work with Philip Dittmann)

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Existential decidability and Hilbert's 10th problem

Let $\mathcal{L}_{\text{ring}}$ always denote the first-order language of rings $\langle +, -, \cdot, 0, 1 \rangle$. All rings are unital and commutative.

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Theorem 1.2 (Davis, Putnam, J. Robinson, Matiyasevich, 1970)

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$\exists$-definability in $\mathbb{Q}$

For a field K with $\mathrm{char}(K) \neq 2$ and $a, b \in K^\times$, define the 2-*fold Pfister form* $\langle\langle a, b\rangle\rangle_K$ as the quadratic form

$$W^2 - aX^2 - bY^2 + abZ^2.$$

For a quadratic form q over K, define the *ramification set* of q as follows:

$$\Delta q = \{v \ \ \mathbb{Z}\text{-valuation on } K \mid q_{K_v} \text{ is anisotropic}\}.$$

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Theorem 1.5 (Poonen, 2009)

There is an $\exists\mathcal{L}_{\mathrm{ring}}$-formula $\psi(x, a, b)$ such that for all $a, b \in \mathbb{Q}^\times$ with $a > 0$

$$\psi(\mathbb{Q}, a, b) = \bigcap_{v \in \Delta\langle\langle a,b\rangle\rangle_{\mathbb{Q}}} \mathcal{O}_v.$$

$\exists$-definability in $\mathbb{Q}$

Corollary 1.6 (Poonen)

There is an $\forall\exists\mathcal{L}_{\mathrm{ring}}$*-formula* $\varphi(x)$ *such that* $\varphi(\mathbb{Q}) = \mathbb{Z}$*. In particular,* $\mathrm{Th}_{\exists\forall\exists}(\mathbb{Q})$ *is undecidable.*

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Theorem 1.8 (Koenigsmann, Park, Eisentraeger-Morrison, D., 2010-2018-)

Let K *be a global field,* R *a finitely generated subring of* K *with* $K = \text{Frac}(R)$. *There is an* $\forall\mathcal{L}_{\text{ring}}(K)$*-formula* $\varphi(x)$ *such that* $\varphi(K) = R$.

$\exists$-defining valuations in (rational) function fields

Theorem 1.9 (Denef, Pheidas, Videla, 1978-1994)

Let K be a field with prime subfield K_0. Suppose that there is an $\exists\mathcal{L}_{\text{ring}}(\{t\})$-formula $\psi(x)$ such that for all $x \in K(t)$ one has

1. $(x \in K_0(t) \wedge v_t(x) > 0) \Rightarrow K(t) \models \psi(x)$,
2. $K(t) \models \psi(x) \Rightarrow v_t(x) > 0$.

Then $\text{Th}_\exists(K(t))$ *is undecidable.*

In particular, $\text{Th}_\exists(K(t))$ is undecidable if the valuation ring at t is existentially definable in $K(t)$.

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- $\mathbb{C}(t_1, t_2)$ (Kim-Roush, 1992).

$\exists$-definability in function fields over global/local fields

Let K be a field, $\mathrm{char}(K) \neq 2$. For $a, b, c \in K^\times$, define the 3-*fold Pfister form* $\langle\langle a, b, c\rangle\rangle_K$ as the quadratic form

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For a field K with $\mathrm{char}(K) \neq 2$, F/K a function field in one variable over K, q a quadratic form defined over F, define

$$\Delta_0 q = \{v \in \Delta q \mid v(K^\times) = 0\}.$$

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Theorem 1.10 (Dittmann, D., 2019-)

Let K be a local or global field, $\mathrm{char}(K) \neq 2$, F a function field in one variable over K. There is an $\exists\mathcal{L}_{\mathrm{ring}}$-formula $\psi(x, a, b, c)$ such that for all $a, b, c \in F^\times$

$$\psi(F, a, b, c) = \bigcap_{v \in \Delta_0 \langle\langle a,b,c \rangle\rangle_F} \mathcal{O}_v.$$

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Corollary 1.11

Let $F = \mathbb{Q}_p(t)$ for some prime number p. The valuation ring

$$\mathcal{O}_{v_t} = \{f \in F \mid v_t(f) \geq 0\}$$

has an $\exists\mathcal{L}_{\text{ring}}(\{t\})$-definition in F.

Proof:

$\exists$-definability in function fields over global/local fields

Theorem 1.12 (Dittmann, D., 2020-)

Let K be a local or global field, $\mathrm{char}(K) \neq 2$, F/K a function field in one variable. For a finitely generated, integrally closed K-subalgebra R of F with $\mathrm{Frac}(R) = F$, there is an $\forall\mathcal{L}_{\mathrm{ring}}(F)$-formula $\varphi(x)$ such that $\varphi(F) = R$.

Outline

Plan for today:

3. Proof of Theorem 1.10 for function fields in one variable over $\mathbb{Q}_p$.

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1. Quadratic forms, local-global principles and existential definability
2. Interactions between valuations and quadratic forms in function fields
3. Proof of Theorem 1.10 for function fields in one variable over $\mathbb{Q}_p$.

Quadratic splitting sets of quadratic forms

Let K be a field, $\mathrm{char}(K) \neq 2$, q a quadratic form over K. Define the *(quadratic) splitting set* of q as

$$S(q) = \{x \in K \mid q_{K[\sqrt{x}]} \text{ is isotropic}\}.$$

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- for a field extension L/K, $S(q) \subseteq S(q_L)$,
- for $a \in S(q)$ we have $aK^{\times 2} \subseteq S(q)$,

Quadratic splitting sets of quadratic forms

- *(local-global principle)* for a collection $\mathcal{E}$ of field extensions of K such that

$$\forall x \in K : (q_{K[\sqrt{x}]} \text{ anisotropic } \Rightarrow \exists E \in \mathcal{E} : q_{E[\sqrt{x}]} \text{ anisotropic})$$

one has

$$S(q) = \bigcap_{E \in \mathcal{E}} S(q_E) \cap K.$$

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E.g. If K is a global field, the Hasse-Minkowski Theorem states that all quadratic forms over K satisfy the required local-global principle if $\mathcal{E}$ is the set of completions of $\mathbb{Z}$-valuations and real closures.

Kato's local-global principle

From now on, $K = \mathbb{Q}_p$ for some prime p, v_p the p-adic valuation on $\mathbb{Q}_p$, F/K a regular function field in one variable. For a quadratic form q defined over F, define

$$\Delta_p q = \{v \in \Delta q \mid v|_K \sim v_p\}.$$

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Theorem 2.1 (Kato, 1986)

Let q be a 3-fold Pfister form defined over F. If $\Delta_p q = \emptyset$, then q is isotropic over F.

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In particular, we obtain for a 3-fold Pfister form q defined over F

$$S(q) = \bigcap_{v \in \Delta_p q} S(q_{F_v}).$$

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Proposition 2.2

Let q be a 3-fold Pfister form over F, $x \in \bigcap_{v \in \Delta_0 q} \mathcal{O}_v$. Then there exists $n \in \mathbb{N}$ such that $p^n x \in \bigcap_{v \in \Delta_p q} \mathfrak{m}_v$.

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For each $v \in \Delta_p q$, one has $p^n x \in \mathfrak{m}_v$ for n sufficiently high. However, $\Delta_p q$ is potentially infinite.

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Proof sketch:

- Using basic results on interactions between valuations and quadratic forms, find a sentence ϕ in the first-order language of valued fields such that for any valuation v on F, $(F, v) \models \phi$ implies $\exists n \in \mathbb{N} : p^n x \in \mathfrak{m}_v$, and such that $(F, v) \models \phi$ whenever $v \in \Delta_p q$.

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- By the Compactness Theorem, there exists an $n \in \mathbb{N}$ such that $p^n x \in \mathfrak{m}_v$ for all $v \in \Delta_p q$.

Parametrised solution

Proposition 3.1

Let q be a 3-fold Pfister form defined over F. Let $d \in S(q) \setminus \{0\}$. For $x \in F$ one has

$$(\forall v \in \Delta_0 q : v(x^2) \geq v(d)) \Leftrightarrow (\exists a \in K^\times : d + (ax)^2 \in S(q)).$$

In particular, if additionally $v(d) = 0$ for all $v \in \Delta_0 q$, then

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As K is existentially definable in F, the property

$$\exists a \in K^\times : d + (ax)^2 \in S(q)$$

is equivalent in F to an existential formula $\psi(x)$.

Parametrised solution

Proof:

Eliminating the parameter d

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- We have shown that there is an existential $\mathcal{L}$-formula φ in 5 free variables such that for all $a, b, c \in F^\times$ and a *good choice* for $d \in F^\times$ we have

$$\bigcap_{v \in \Delta_0 \langle\langle a,b,c \rangle\rangle} \mathcal{O}_v = \{x \in F^\times \mid F \models \varphi(x, a, b, c, d)\}.$$

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- We would like to get rid of the need to choose an appropriate $d \in F^\times$ (this hinders universal quantification over the a, b, c).

Eliminating the parameter d

Proposition 3.3

Let q be a three-fold Pfister form defined over F. Then

$$\bigcap_{v\in\Delta_0 q}\mathcal{O}_v=\left\{x\in F\;\middle|\;\begin{array}{l}\exists e\in S(q)\setminus\{0,1\},a\in K^\times:\\ \frac{e}{(e-1)^2}+(ax)^2\in S(q)\end{array}\right\}.$$

Proof: