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A universal definition of $\mathbb{Z}$ in $\mathbb{Q}$

Nicolas Daans

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Existentially definable sets

Let K be a field, $D \subseteq K^n$ for some $n \in \mathbb{N}$.

D is called *existentially definable (in K^n)* if there exist $r, m \in \mathbb{N}$ and polynomials $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ such that

$$D = \{x \in K^n \mid \exists y \in K^m : f_1(x, y) = \ldots = f_r(x, y) = 0\}.$$

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- For $K = \mathbb{Q}$, the set of non-negative rationals:

$$\mathbb{Q}_{\geq 0} = \{x \in \mathbb{Q} \mid \exists y_1, \ldots, y_4 \in \mathbb{Q} : x = y_1^2 + y_2^2 + y_3^2 + y_4^2\}.$$

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- (essentially by J. Robinson, [Rob49]) For $K = \mathbb{Q}$ and a prime number p, the subring $\mathbb{Z}_{(p)} = \{\frac{x}{y} \mid x \in \mathbb{Z}, y \in \mathbb{Z} \setminus p\mathbb{Z}\}$. E.g. if $p \equiv 3 \bmod 4$, then

$$\mathbb{Z}_{(p)} = \{x \in \mathbb{Q} \mid \exists y_1, y_2, y_3 \in \mathbb{Q} : 1+(p-1)px^2 = y_1^2+y_2^2+py_3^2\}.$$

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- If $D \subseteq K^{n+m}$ is existentially definable and $E \subseteq K^m$ is existentially definable, then

$$\bigcup_{y \in E} \{x \in K^n \mid (x,y) \in D\} = \{x \in K^n \mid \exists y \in E : (x,y) \in D\}$$

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  is existentially definable in K^n.
- If K is not algebraically closed, then every existentially definable subset of K^n is of the form

$$\{x \in K^n \mid \exists y \in K^m : f(x, y) = 0\}$$

  for some $m \in \mathbb{N}$, $f \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$.

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- If K has a "rich arithmetic" (e.g. number field), many subsets could be existentially definable.

Question 1.1

Is $\mathbb{Z}$ existentially definable in $\mathbb{Q}$?

Existentially definable sets

Let $n \in \mathbb{N}$. A subset D of K^n is called *universally definable (in K^n)* if $K^n \setminus D$ is existentially definable in K^n. In other words, D is universally definable if there exist $r, m \in \mathbb{N}$ and polynomials $f_1, \ldots, f_r \in K[X_1, \ldots, X_n, Y_1, \ldots, Y_m]$ such that

$$D = \{x \in K^n \mid \forall y \in K^m : f_1(x, y) \neq 0 \text{ or } \ldots \text{ or } f_r(x, y) \neq 0\}.$$

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Theorem 1.2 (Koenigsmann, 2016)

$\mathbb{Z}$ is universally definable in $\mathbb{Q}$.

Theorem 1.3 (Park, 2013)

Let K be a number field. Denote by $\mathcal{O}_K$ the ring of integers of K. Then $\mathcal{O}_K$ is universally definable in K.

Existentially definable sets

Let K be a global field (number field or function field in one variable over a finite field).
Denote by $\mathcal{V}(K)$ the set of all $\mathbb{Z}$-valuations on K. For a finite subset $S \subseteq \mathcal{V}(K)$, define the *ring of S-integers* of K to be

$$\mathcal{O}_S = \bigcap_{v \in \mathcal{V}(K) \setminus S} \mathcal{O}_v.$$

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Theorem 1.4 (Eisenträger-Morrison, 2018)

Let K be a global function field of odd characteristic. For any finite $S \subseteq \mathcal{V}(K)$, $\mathcal{O}_S$ is universally definable in K.

E.g. $\mathbb{F}_q[T]$ is universally definable in $\mathbb{F}_q(T)$.

Outline

Plan for the rest of the talk:

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Plan for the rest of the talk:

- Explain how (properties of) quaternion algebras over global and local fields play a role, building on ideas of Poonen & Koenigsmann.
- Give a proof of Koenigsmann's Theorem (universal definability of $\mathbb{Z}$ in $\mathbb{Q}$).
- Discuss generalisations to other global fields and function fields

The ramification set

Denote by $\mathbb{P}$ the set of prime numbers, and set $\mathbb{P}' = \mathbb{P} \cup \{\infty\}$. Define $\mathbb{Q}_\infty = \mathbb{R}$.

For $a, b \in \mathbb{Q}^\times$, define the *ramification set* of the quaternion algebra $(a, b)_\mathbb{Q}$:

$$\Delta(a, b) = \{p \in \mathbb{P}' \mid (a, b)_{\mathbb{Q}_p} \text{ is non-split}\}.$$

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Recall: $(a, b)_\mathbb{Q} \cong (ac^2, bd^2)_\mathbb{Q}$ for $a, b, c, d \in \mathbb{Q}^\times$, whence $\Delta(a, b) = \Delta(ac^2, bd^2)$.

The ramification set

The ramification set can be computed precisely as follows:

Proposition 2.1 (Computation of the ramification set)

Let $a, b \in \mathbb{Z} \setminus \{0\}$ be square-free.

1. $\infty \in \Delta(a,b)$ *if and only if* $a < 0$ *and* $b < 0$.
2. *For* $p \in \mathbb{P} \setminus \{2\}$ *we have* $p \in \Delta(a,b)$ *if and only if one of the following holds*
   - $p \mid a$, $p \nmid b$, *and* b *is not a square mod* p
   - $p \mid b$, $p \nmid a$, *and* a *is not a square mod* p
   - $p \mid a$, $p \mid b$, *and* $-abp^{-2}$ *is not a square mod* p
3. *If* $2 \mid b$ *and* $a \equiv 5 \bmod 8$, *then* $2 \in \Delta(a,b)$.
4. *(Hilbert Reciprocity)* $|\Delta(a,b)|$ *is an even natural number.*

Note: we can scale any $a, b \in \mathbb{Q}^\times$ by a square to obtain square-free elements of $\mathbb{Z} \setminus \{0\}$.

The ramification set

Lemma 2.2

Let p, q be odd prime numbers such that $q \equiv 5 \bmod 8$ and q is not a square modulo p. We have:

$$\Delta(q, 2p) = \{2, p\}.$$

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Corollary 2.3

For every odd prime number p we can find $a \in \mathbb{Z}_{(2)}^{\times}$ such that $\Delta(1 + 4a^2, 2p) = \{2, p\}$.

Existentially definable semilocal building blocks

For $a, b, c \in \mathbb{Q}^\times$, define

$$\Delta^c(a,b) = \{p \in \Delta(a,b) \cap \mathbb{P} \mid v_p(c) \text{ is odd}\}$$

and for $a, b, c \in \mathbb{Q}^\times$, set

$$J^c(a,b) = \bigcap_{p \in \Delta^c(a,b)} p\mathbb{Z}_{(p)} \subseteq \mathbb{Q}.$$

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Theorem 2.4

There exists an existentially definable set $D \subseteq \mathbb{Q}^4$ such that, for all $a, b, c \in \mathbb{Q}^\times$, we have

$$J^c(a, b) = \{x \in \mathbb{Q} \mid (x, a, b, c) \in D\}$$

Relies on work by Poonen, Koenigsmann.

Existential to universal

The following observation (implicit in Koenigsmann's work) links uniform existential definability of prime ideals with universal definability.

Lemma 3.1

If $\bigcup_{p \in \mathbb{P}} p\mathbb{Z}_{(p)}$ is existentially definable in $\mathbb{Q}$, then $\mathbb{Z}$ is universally definable in $\mathbb{Q}$.

Proof of main theorem

Proposition 3.2

Setting

$$\Phi = \{(1+4a^2, 2b) \mid a, b \in \mathbb{Z}_{(2)}^{\times}\}$$

we have

$$\bigcup_{p\in\mathbb{P}} p\mathbb{Z}_{(p)} = \left(\bigcup_{(x,y)\in\Phi} J^x(x,y) \cap J^{2y}(x,y)\right) \cup 2\mathbb{Z}_{(2)}.$$

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Since the set on the right is existentially definable, so is the set on the left. The universal definability of $\mathbb{Z}$ in $\mathbb{Q}$ now follows from the previous lemma.

Quantitative aspects

We have shown that there exist $m \in \mathbb{N}$ and $f \in \mathbb{Q}[X, Y_1, \ldots, Y_m]$ such that

$$\mathbb{Z} = \{x \in \mathbb{Q} \mid \forall y \in \mathbb{Q}^m : f(x, y) \neq 0\}.$$

What is the value of m in this description? How low can it be chosen?

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- $m \leq 1$ is not possible.

Generalisation to all global fields

The approach extends to general global fields (i.e. number fields or function fields in one variable over a finite field). One needs to replace concepts adequately:

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Theorem 4.1

Let K be a global field. For any finite $S \subseteq \mathcal{V}(K)$, $\mathcal{O}_S$ is universally definable in K.

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- Rings of S-integers of a global field K are precisely integrally closed, finitely generated subrings of K with K as fraction field.

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- Rings of S-integers of a global field K are precisely integrally closed, finitely generated subrings of K with K as fraction field.
- In fact, one can obtain the following (suggested by Dittmann):

Corollary 4.2

If R is a finitely generated domain with a global field K as fraction field, then R is universally definable in K.

Abstraction of the question

We used that $\mathbb{Z} = \bigcap_{p \in \mathbb{P}} \mathbb{Z}_{(p)}$, i.e. $\mathbb{Z}$ is the intersection of all discrete valuations rings of $\mathbb{Q}$.

Question 4.3

Given a field K, which intersections of valuation rings can we define existentially/universally?

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Given a field K, which intersections of valuation rings can we define existentially/universally?

There are two key ingredients in our proof:

- Existential definability of $\bigcap_{v \in \Delta^c(a,b)} \mathfrak{m}_v$ uniformly in a, b, c.
- Good description of ramification sets, in particular a Reciprocity Law

Function fields in one variable over other fields

(joint work with Philip Dittmann)

Let F be a function field in one variable over a field K (let's say $\mathrm{char}(K) \neq 2$ for now). Let $\mathcal{V}$ be the set of $\mathbb{Z}$-valuations which are trivial on K. Under which conditions can we still universally define rings of S-integers?

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- If K is a finite field, then K has 2-cohomological dimension 1, and a quaternion algebra $(a, b)_F$ can be seen as an element of $H^2(F, \mathbb{Z}/2\mathbb{Z})$. If K is a field of 2-cohomological dimension n, one should work with elements of $H^{n+1}(F, \mathbb{Z}/2\mathbb{Z})$. $H^{n+1}(F, \mathbb{Z}/2\mathbb{Z})$ is generated by symbols $(a_1, \ldots, a_{n+1})_F$ with $a_1, \ldots a_{n+1} \in F^\times$. We define $\Delta(a_1, \ldots, a_{n+1})$ and $\Delta^c(a_1, \ldots, a_{n+1})$ analogously.

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- There should be an existential formula associating to $(a_1, \ldots, a_{n+1}, d) \in (F^\times)^{n+2}$ the subset $\bigcap_{v \in \Delta^d(a_1,\ldots,a_{n+1})} \mathfrak{m}_v$.

Function fields in one variable over other fields

If F/K a function field in one variable and K is of 2-cohomological dimension n, then the natural sequence

$$H^{n+1}(F, \mathbb{Z}/2\mathbb{Z}) \to \bigoplus_{v \in \mathcal{V}} H^n(Fv, \mathbb{Z}/2\mathbb{Z}) \to H^n(K, \mathbb{Z}/2\mathbb{Z}) \to 0$$

is exact. If additionally all elements of $H^{n+1}(F, \mathbb{Z}/2\mathbb{Z})$ are symbols $(a_1, \ldots, a_{n+1})_F$, then we obtain a sort of Reciprocity Law for these symbols.

Function fields in one variable over other fields

Some examples where this works:

Theorem 4.4

Let K be a field with $\mathrm{char}(K) \neq 2$ satisfying one of the following:

- *K is a global field,*
- *K is a local field, i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((T))$ for some prime p,*
- *K is a pseudo-algebraically closed, non-quadratically closed field.*

Then for any function field in one variable F/K and any finite set S of $\mathbb{Z}$-valuations on F trivial on K, the ring of S-integers $\mathcal{O}_S$ is universally definable.

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Corollary 4.5

Let K be a field with $\mathrm{char}(K) \neq 2$ satisfying one of the following:

- *K is a local field,*
- *K is a pseudo-algebraically closed, non-quadratically closed field.*

Let R be a domain which is finitely generated as a K-algebra and whose fraction field F is a function field in one variable over K. Then R is universally definable in F.

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