Polynomially presented properties: preservation proposition perspectives

CUNI MFF Algebra Colloquium

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24 February 2026

Properties described by equations

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"7 is twice the cube of an element"
"-1 is a sum of 3 squares"
"There is an element other than 0 which is both a square and minus a square"

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We can also consider properties which an element X or tuple of elements $(X_1, \ldots, X_n)$ of the field may or may not have.

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	$\exists Y : (X - 2)(X - Y^2) = 0$

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  $\rightarrow$ is equivalent to $\exists Y_1, Y_2 : -1 = Y_1^2 + Y_2^2$, so needs only 2 variables!
  (**Proof:** If $-1 = y_1^2 + y_2^2 + y_3^2$, then either $-1 = y_1^2$, or $-1 = \left(\frac{y_2+y_1y_3}{1+y_1^2}\right)^2 + \left(\frac{y_3-y_1y_2}{1+y_1^2}\right)^2$.)

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- $\exists Y_1, Y_2, Y_3 : X = Y_1^2 + Y_2^2 + Y_3^2$ maybe needs 3 variables ?
- $\exists Y_1, Y_2, Y_3 : X_1 = Y_1^2, X_2 = Y_2^2, X_3 = Y_3^2$ maybe needs 3 variables ?

Existential rank

Semi-formally:

Question

Let $\mathcal{K}$ be a class of fields, $n \in \mathbb{N}$. Let $\mathcal{P}(X_1, \ldots, X_n)$ be a property which a n-tuple of elements $(x_1, \ldots, x_n)$ in a field $K \in \mathcal{K}$ may or may not have.
What is the smallest $m \in \mathbb{N}$ (if it exists) for which the property $\mathcal{P}(X_1, \ldots, X_n)$ is equivalent to the solvability of $\mathcal{S}(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ in K (for $Y_1, \ldots, Y_m$) for all $K \in \mathcal{K}$, where $\mathcal{S}$ is some boolean combination of polynomial equalities?

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The answer to the above question will be called the existential rank of $\mathcal{P}$ (with respect to $\mathcal{K}$).
E.g. With respect to the class of all fields, the property "-1 is a sum of 3 squares" has existential rank at most 2.

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Proposition

Let K be a field, $\mathrm{char}(K) \neq 2$, $x_1, x_2 \in K$. Then the following are equivalent:

1. $\exists y_1, y_2 \in K : x_1 = y_1^2$ *and* $x_2 = y_2^2$,
2. $\exists z \in K : (x_1 = x_2 = 0)$ *or* $(z^4(x_1 - x_2)^2 - 2z^2(x_1 + x_2) = 1)$.

In particular, the existential rank of "X_1 and X_2 are both squares" with respect to $\mathcal{K}$ is 1.

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Proof.

(1) $\Rightarrow$ (2): If $x_1 \neq 0$ or $x_2 \neq 0$, then without loss of generality $y_1 + y_2 \neq 0$. Set $z = \frac{1}{y_1+y_2}$.

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(2) $\Rightarrow$ (1): If $x_1 \neq 0$ or $x_2 \neq 0$, then $z \neq 0$. Set $y_1 = \frac{1+z^2(x_1-x_2)}{2z}$ and $y_2 = \frac{1+z^2(x_2-x_1)}{2z}$. $\square$

Outline

1. Introduction: optimally representing properties by solvability of polynomial equations ✓
2. Crash course: Preservation theorems in model theory and algebra
3. Existential rank revisited
4. Outlook & a lot of open questions

First-order structures

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An $\mathcal{L}$-structure $\mathcal{A}$ consists of a set A, and

- for every constant symbol c, an element $c^{\mathcal{A}} \in A$,
- for every n-ary function symbol f, a function $f^{\mathcal{A}} : A^n \to A$,
- for every n-ary relation symbol R, a set $R^{\mathcal{A}} \subseteq A^n$.

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Main example for this presentation: the <u>language of rings</u> $\mathcal{L}_{\text{ring}}$ consists of:

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Note: not every $\mathcal{L}_{\text{ring}}$-structure is a ring.

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Example: an $\mathcal{L}_{\text{ring}}$-ring structure $\mathcal{A}$ is a ring if and only if the following sentences are satisfied:

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\begin{aligned}
&\forall X, Y, Z((X+Y)+Z = X+(Y+Z) \wedge (X \cdot Y) \cdot Z = X \cdot (Y \cdot Z)),\\
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&\forall X(X \cdot 1 = X \wedge 1 \cdot X = X \wedge X + 0 = X),\\
&\forall W, X, Y, Z((W+X) \cdot (Y+Z) = ((W \cdot Y + W \cdot Z) + X \cdot Y) + X \cdot Z),\\
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The ring is a field if it additionally satisfies

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A set of $\mathcal{L}$-sentences is called an $\mathcal{L}$-theory. An $\mathcal{L}$-structure in which all sentences of a given $\mathcal{L}$-theory T holds is called a model of T. E.g. An $\mathcal{L}_{\text{ring}}$-structure $\mathcal{A}$ is a field if and only if it is a model of the theory T consisting of all sentences on this slide.

∃-formulas and ∀-formulas

$\mathcal{L}$-sentences of the respective forms

$$\forall Y_1, Y_2, \ldots, Y_m \psi(Y_1, \ldots, Y_m) \quad \text{and} \quad \exists Y_1, \ldots, Y_m \psi(Y_1, \ldots, Y_m)$$

where $\psi(Y_1, \ldots, Y_m)$ is a Boolean combination of atomic formulas (in the case of $\mathcal{L}_{\text{ring}}$: polynomial equalities), are called universal $\mathcal{L}$-formulas (∀-formulas) and existential $\mathcal{L}$-formulas (∃-formulas) respectively.
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A $\forall$-theory (respectively $\exists$-theory) is a theory consisting only of $\forall$-sentences (respectively $\exists$-sentences).

How does one recognize an $\forall$-theory ($\exists$-theory)?

$\exists$-formulas and $\forall$-formulas

Definition: When $\mathcal{A}$ is an $\mathcal{L}$-structure on set A, and if $B \subseteq A$ contains all constants $c^{\mathcal{A}}$ and is closed under all functions $f^{\mathcal{A}}$, then there is a natural induced $\mathcal{L}$-structure $\mathcal{B}$ on B via restriction; we call $\mathcal{B}$ an <u>$\mathcal{L}$-substructure</u> of $\mathcal{A}$ and write $\mathcal{B} \subseteq \mathcal{A}$.

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Proposition

Every $\mathcal{L}_{\text{ring}}$-substructure of a ring is a ring.

Proof.

All $\mathcal{L}_{\text{ring}}$-sentences (axioms) for rings are $\forall$-sentences. When they hold in a structure, then also in every substructure. $\square$

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Proposition

The class of $\mathcal{L}_{\text{ring}}$-structures which are fields can not be described by an $\forall$-theory.

Proof.

$\mathbb{Q}$ is a field, but its $\mathcal{L}_{\text{ring}}$-substructure $\mathbb{Z}$ is not. □

$\exists$-formulas and $\forall$-formulas

Theorem (Łoś-Tarski Preservation Theorem ($\forall$-version), 1954)

Let T be an $\mathcal{L}$-theory. Suppose that every substructure of a model of T is again a model of T. Then T is equivalent to an $\forall$-theory.

(Call two theories equivalent if they have the same models.)

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Example: Add to the theory of rings the $\mathcal{L}$-sentence

$$\exists X_1, X_2, X_3, X_4 \forall Y (X_1 = Y \vee X_2 = Y \vee X_3 = Y \vee X_4 = Y).$$

This axiom expresses precisely that the ring has at most 4 elements.

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For example (by Pigeonhole Pricniple):

$$\forall Y_1, Y_2, Y_3, Y_4, Y_5 \left(\bigvee_{1 \leq i < j \leq 5} (Y_i = Y_j) \right)$$

Another example: $\forall\exists$-formulas

$\mathcal{L}$-sentences of the form

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Definition: When $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \ldots$ is an ascending chain of $\mathcal{L}$-structures with underlying sets $A_0 \subseteq A_1 \subseteq A_2 \subseteq \ldots$, the union $\bigcup_{i\in\mathbb{N}} A_i$ naturally carries an $\mathcal{L}$-structure, denoted $\bigcup_{i\in\mathbb{N}} \mathcal{A}_i$ and called the directed union.

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Proposition

Let $K_0 \subseteq K_1 \subseteq \dots$ be an ascending chain of fields. Then $\cup_{i \in \mathbb{N}} K_i$ is also a field.

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All $\mathcal{L}_{\text{ring}}$-sentences (axioms) for fields are $\forall\exists$-sentences. When they hold in an ascending chain of structures, then also in their union. $\square$

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Theorem (Chang-Łoś-Suszko Preservation Theorem, 1960's)

Let T be an $\mathcal{L}$-theory. Suppose that, whenever $\mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$ is a chain of models of T, also $\bigcup_{i\in\mathbb{N}} \mathcal{A}_i$ is a model of T. Then T is equivalent to an $\forall\exists$-theory.

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1. Introduction: optimally representing properties by solvability of polynomial equations ✓
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3. Existential rank revisited
4. Outlook & a lot of open questions

Recall: existential rank

Recall the original question:

Question

Let $\mathcal{K}$ be a class of fields, $n \in \mathbb{N}$. Let $\mathcal{P}(X_1, \ldots, X_n)$ be a property which a n-tuple of elements $(x_1, \ldots, x_n)$ in a field $K \in \mathcal{K}$ may or may not have.
What is the smallest $m \in \mathbb{N}$ (if it exists) for which the property $\mathcal{P}(X_1, \ldots, X_n)$ is equivalent to the solvability of $\mathcal{S}(X_1, \ldots, X_n, Y_1, \ldots, Y_m)$ in K (for $Y_1, \ldots, Y_m$) for all $K \in \mathcal{K}$, where $\mathcal{S}$ is some boolean combination of polynomial equalities?

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The answer to the above question will be called the existential rank of $\mathcal{P}$ (with respect to $\mathcal{K}$).

We are looking for a criterion for a property $\mathcal{P}$ to be given as

$$\exists Y_1, \ldots, Y_m \psi$$

where ψ is a Boolean combination of polynomial equations.

The property "$X_1, \ldots, X_n$ are squares"

Consider the property $\pi_n(X_1, \ldots, X_n)$ stating that $X_1, \ldots, X_n$ are all squares, i.e. given by

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Let $\mathcal{K}$ be the class of all fields, $n \in \mathbb{N}$. Then π_n has existential rank n with respect to $\mathcal{K}$.

Proof on blackboard.

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For contrast, we have:

Proposition

Let $\mathcal{K}$ be the class of all fields K with $\mathrm{char}(K) \neq 2$, let $n \geq 1$. Then π_n has existential rank 1 with respect to $\mathcal{K}$.

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Proof idea.

1. If K is any field with $\mathrm{char}(K) \neq 2$ and $a_1, \ldots, a_n \in K$, the extension $K(\sqrt{a_1}, \ldots, \sqrt{a_n})/K$ is generated by a single element by the *Primitive Element Theorem*.

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Note: It is possible to find an explicit formula with 1 existential quantifier for all n (Becher, D., ca. 2019, only in my PhD thesis).

Existential rank and essential dimension

In “Existential rank and essential dimension of diophantine sets” (D., Dittmann, Fehm, 2021) we develop systematically the connection between two related concepts of complexity:

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- Existential rank of properties in fields,
- essential and canonical dimension of varieties (as introduced by Merkurjev, Berhuy, Favi, 2003 - based on earlier work of Buhler and Reichstein, 1997).
  Informally: the essential dimension of an algebraic object is the minimal number of algebraically independent parameters needed to describe it.

Existential rank and essential dimension

For example, by using results of Karpenko and Merkurjev (2003) which imply that the projective quadratic over $\mathbb{Q}(T)$ given by the equation

$$TX_0^2 = X_1^2 + \ldots + X_n^2$$

has canonical dimension $n-1$, we obtain:

Proposition

Let $\mathcal{K}$ be the class of all fields of characteristic 0, $n \in \mathbb{N}$. Then the formula

$$\exists Y_1, \ldots, Y_n : X = Y_1^2 + \ldots + Y_n^2$$

has existential rank n.

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Thanks for your attention!

[BF03] Grégory Berhuy and Giordano Favi. "Essential Dimension: A functorial point of view (after A. Merkurjev)". In: Documenta Mathematica 8 (2003), pp. 279–330.

[Daa22] Nicolas Daans. "Existential first-order definitions and quadratic forms". Available as hdl.handle.net/10067/1903760151162165141. PhD thesis. University of Antwerp, 2022.

[DDF21] Nicolas Daans, Philip Dittmann, and Arno Fehm. "Existential rank and essential dimension of diophantine sets". Available as arXiv:2102.06941. 2021.

[KM03] Nikita Karpenko and Alexander Merkurjev. "Essential dimension of quadrics". In: Inventiones Mathematicae 153.2 (2003), pp. 361–372.

[Pas22] Hector Pasten. "Notes on the DPRM property for listable structures". In: The Journal of Symbolic Logic 87.1 (2022), pp. 273–312.

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